AN UPPER BOUND OF REGULARITY OF EDGE IDEALS

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Abstract: Let G be a simple graph. We give an upper bound for $\operatorname{reg} I(G)$ in terms of the induced matching number of its spanning trees.

1 Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Let G be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. One associate to G a quadratic square-free monomial ideal

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \text{ in } R,$$

which is called the *edge ideal* of G.

The *Castelnuovo-Mumford regularity* (or *regularity* for short) of an edge ideal of a finite simple graph has been studied in many articles including [1, 2, 4, 8, 10, 11, 12].

A set $\mathcal{M} \subseteq E(G)$ is a *matching* of G if two different edges in $\mathcal{M}$ are disjoint; and the *matching number* of G, denoted by $\nu(G)$, is defined by

$$\nu(G) := \max\{|\mathcal{M}| \mid \mathcal{M} \text{ is a matching of } G\}.$$

A set $\mathcal{M} = \{a_1b_1, \ldots, a_rb_r\} \subseteq E(G)$ is an *induced matching* of G if the induced subgraph of G on the vertex set $\{a_1, b_1, \ldots, a_r, b_r\}$ consists of just r disjoint edges; and the *induced matching number* of G, denoted by $\nu_0(G)$, is defined by

$$\nu_0(G) := \max\{|\mathcal{M}| \mid \mathcal{M} \text{ is an induced matching of } G\}.$$

Then, the basic inequalities that relate $\operatorname{reg} I(G)$ to the matching number and the induced matching number of G are

$$\nu_0(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 1,$$

where the first inequality is proved by Katzman [10] and the second one is proved by Hà and Van Tuyl [8].

The aim of this paper is to give another upper bound of $\operatorname{reg} I(G)$ in terms of *spanning trees* of G. This result is an improvement of the second inequality above. Recall that a

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spanning tree T of a *connected* graph G is a subgraph of G that is a tree which includes all of the vertices of G. The main result of the paper is the following theorem.

Theorem 2.5. *Let G be a connected graph. Then,*

$$\operatorname{reg} I(G) \leqslant \max\{\nu_0(T) + 1 \mid T \textit{ is a spanning tree of } G\}.$$

2 The proof of the result

Let k be a field, and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over k with n variables. The object of our work is the regularity of graded modules and ideals over R. This invariant can be defined in various ways. In this paper we recall the definition that uses the minimal free resolution (see [5]). Let M be a finitely generated graded R-module, and let

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be its minimal free resolution.

For each i, let $t_i(M)$ be the largest degree of a system of minimal homogeneous generators of F_i. Then, the regularity of M is defined by

$$\operatorname{reg} M = \max\{t_i(M) - i \mid i = 0, \ldots, p\}.$$

Next we recall some terminologies from the Graph theory (see [3]). Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The union of G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We use the symbol $v(G)$ to denote $|V(G)|$ and the symbol $\varepsilon(G)$ do denote $|E(G)|$.

A *path* in G is an alternating sequence of vertices and edges

$$u_1, e_1, u_2, e_2, ..., e_{m-1}, u_m,$$

in which $e_i = \{u_i, u_{i+1}\}$. We say that this path is of length $m - 1$ and is from u_1 to u_m. The graph G connected if there is a path from any vertex to any other vertex in the graph. If G is not connected, then it is a disjoint union of its connected subgraphs; each such a connected graph is called a connected component of G.

For a vertex u in G, let $N_G(u) = \{v \in V(G) \mid \{u, v\} \in E(G)\}$ be the set of neighbors of u. An edge e is incident to a vertex u if $u \in e$. The degree of a vertex $u \in V(G)$, denoted by $\deg_G(u)$, is the number $|N_G(u)|$. If $\deg u = 0$, then u is called an *isolated* vertex of G. If every vertex of G is isolated, then G is called a *totally disconnected* graph. For an edge e in G, define $G \setminus e$ to be the subgraph of G with the edge e deleted (but its vertices remained). For a subset $W \subseteq V(G)$, define $G[W]$ to be the subgraph of G with the vertices in W (and their incident edges) deleted. If $e = \{u, v\}$, then define G_e to be the induced subgraph $G[V(G) \setminus (N_G(u) \cup N_G(v))]$ of G.

Example 2.1. Let G be the cycle C_6 as in the Figure 1.

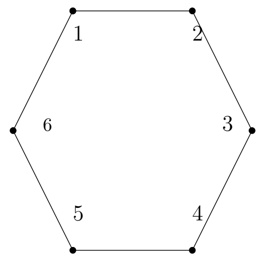

Figure 1: *The cycle* C_6

Then we have $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1)$. Let e be the edge $\{1, 6\}$. Then, $G \setminus e$ is the path of length 5 that goes through 1 to 6; and the graph G_e is just the edge $\{3, 4\}$. Note that $\nu_0(G) = 2$ and $\nu(G) = 3$.

By using a computer program Macaulay2 (see [6]) we get $\operatorname{reg} I(G) = 3$.

In the study on the regularity of edge ideals, induction has proved to be a powerful technique. In the proof of our theorem we use the following results.

Lemma 2.2. *[7, Lemma 3.1 and Theorem 3.5] Let G be a graph. Then,*

1. *If H is an induced subgraph of G, then $\operatorname{reg} I(H) \leqslant \operatorname{reg} I(G)$.*
2. *If e is an edge of G, then*

$$\operatorname{reg} I(G) \leqslant \max\{2, \operatorname{reg} I(G \setminus e), \operatorname{reg}(G_e) + 1\}.$$

Lemma 2.3. *Let G be a graph with connected components $G_1, \ldots, G_s$. Then,*

$$\operatorname{reg} I(G) = \sum_{i=1}^{s} \operatorname{reg} I(G_i) - s + 1.$$

Proof. Since $I(G) = I(G_1) + \cdots + I(G_s)$, the lemma follows from [9, Corollary 2.4].

If every connected component of a graph is a tree, then it is called a *forest*.

Lemma 2.4 (12, Theorem 2.18)**.** *If G is a forest, then* $\operatorname{reg} I(G) = \nu_0(G) + 1$.

We now are in position to prove the main result of this paper.

Theorem 2.5. *Let G be a connected graph. Then,*

$$\operatorname{reg} I(G) \leqslant \max\{\nu_0(T) + 1 \mid T \textit{ is a spanning tree of } G\}$$

Proof. We prove the theorem by induction on $m(G) := v(G) + \varepsilon(G)$. If G is totally disconnected, then it is just one vertex, and then the theorem holds. Assume that G is not totally disconnected. Note then that $m(G) \geqslant 3$.

If $m(G) = 3$, then G is just one edge, and then the theorem holds true.

Assume that $m(G) > 3$. If G is a tree, then the theorem follows from Lemma 2.4. Thus, we assume that G is not a tree. Let e be an edge lying in a cycle of G. Then, $G \setminus e$ is still connected. Now we consider two cases:

Case 1: $\operatorname{reg} I(G) \leqslant \operatorname{reg} I(G \setminus e)$. Since $v(G \setminus e) = v(G)$ and $\varepsilon(G) = \varepsilon(G \setminus e) + 1$, we have $m(G \setminus e) = m(G) - 1$. By the induction hypothesis, $G \setminus e$ has a spanning tree T such that $\operatorname{reg} I(G \setminus e) \leqslant \nu_0(T) + 1$. Hence, $\operatorname{reg} I(G) \leqslant \operatorname{reg} I(G \setminus e) \leqslant \nu_0(T) + 1$.

Note that T is a spanning tree of G as well, so the theorem holds for this case.

Case 2: $\operatorname{reg} I(G) > \operatorname{reg} I(G \setminus e)$. By Lemma 2.2(2) we have

$$\operatorname{reg} I(G) \leqslant \max\{\operatorname{reg} I(G \setminus e), \operatorname{reg} I(G_e) + 1\}.$$

Thus, $\operatorname{reg} I(G) \leqslant \operatorname{reg} I(G_e) + 1$.

Let $G_1, \ldots, G_s$ be connected components of $G \setminus e$. Since each G_i is an subgraph of G, $m(G_i) < m(G)$.

Now for every $i = 1, \ldots, s$, by the induction, there is a spanning tree T_i of G_i such that

$$\operatorname{reg} I(G_i) \leqslant \nu_0(T_i) + 1. \tag{1}$$

For simplicity, let T_0 be the tree with only edge e so that $\nu_0(T_0) = 1$. Then, $T_0, T_1, \ldots, T_s$ are subtrees of G with disjoint vertex sets, so there is a spanning tree of G such that $T_0, T_1, \ldots, T_s$ are its induced subgraphs. Note that for $i \neq j$, there are no edges in G that connect some vertex of T_i to another one of T_j. Thus, any union of induced matchings of $T_0, T_1, \ldots, T_s$ is also an induced matching of T. In particular,

$$\nu_0(T) \geqslant \nu_0(T_0) + \nu_0(T_1) + \cdots + \nu_0(T_s) = 1 + \sum_{i=1}^{s} \nu_0(T_i).$$

Together with Lemma 2.3, we obtain

$$\begin{aligned} \operatorname{reg} I(G_e) &= \left(\sum_{i=1}^{s} \operatorname{reg} I(G_i)\right) - s + 1 \leqslant \sum_{i=1}^{s}(\nu_0(T_i) + 1) - s + 1 \\ &\leqslant \sum_{i=1}^{s} \nu_0(T_i) + 1 \leqslant (\nu_0(T) - 1) + 1 = \nu_0(T). \end{aligned}$$

Therefore, $\operatorname{reg} I(G) \leqslant \operatorname{reg} I(G_e) + 1 \leqslant \nu_0(T) + 1$, and the proof of the theorem now is complete.

As a consequence, we recover a result of Hà and Van Tuyl [8] (see [8, Theorem 6.7]).

Corollary 2.6. $\operatorname{reg} I(G) \leqslant \nu(G) + 1$.

Proof. By Theorem 2.5, there is a spanning tree of G such that $\operatorname{reg} I(G) \leqslant \nu_0(T) + 1$. Since any matching of T is a matching of G, we have $\nu(T) \leqslant \nu(G)$. Thus,

$$\operatorname{reg} I(G) \leqslant \nu_0(T) + 1 \leqslant \nu(T) + 1 \leqslant \nu(G) + 1.$$

A connected graph G is called a *unicyclic* graph if it has only one cycle. For such a graph, Biyikoğlu and Civan proved that $\operatorname{reg} I(G) \leqslant \nu_0(G) + 2$ (see [2, Corollary 4.12]).

Note that for any connected graph G we have $\upsilon(G) \leqslant \varepsilon(G) + 1$. Moreover, $\upsilon(G) = \varepsilon(G)+1$ if and only if G is a tree, $\upsilon(G) = \varepsilon(G)$ if and only if G is unicyclic, and $\upsilon(G) < \varepsilon(G)$ in other cases. By using Theorem 2.5 we can generalize the result of Biyikoğlu and Civan as follows.

Proposition 2.7. *Let G be a connected graph. Then,*

$$\operatorname{reg} I(G) \leqslant \nu_0(G) + \varepsilon(G) - \upsilon(G) + 2.$$

Proof. We first prove the following claim:

Claim: For any connected graph H and an edge e of H such that $H \setminus e$ is connected, we have

$$\nu_0(H \setminus e) \leqslant \nu_0(H) + 1.$$

Indeed, let $\{e_1, \ldots, e_r\}$, where $r = \nu_0(H \setminus e)$, be an induced matching of $H \setminus e$. If $e_i \cap e = \emptyset$ for each i, then $\{e_1, \ldots, e_s\}$ is an induced matching of G. This implies $\nu_0(H) \geqslant r = \nu_0(H)$.

If $e \cap e_i \neq \emptyset$ for some i, we may assume that $i = r$. Then we can verify that $\{e_1, \ldots, e_{r-1}\}$ is an induced matching of H. This implies $\nu_0(H) \geqslant r - 1 = \nu_0(H \setminus e) - 1$, and the claim follows.

We now turn to prove the proposition. By Theorem 2.5 there is a spanning tree T of G such that $\operatorname{reg} I(G) \leqslant \nu_0(T) + 1$. Let $r = \varepsilon(G) - \upsilon(G) + 1$. In order to prove the proposition it suffices to show that $\nu_0(T) \leqslant \nu_0(G) + r$.

Since the tree T is obtained from G by deleting r edges from G, and hence the inequality $\nu_0(T) \leqslant \nu_0(G) + r$ follows from the claim above by induction on r.

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TÓM TẮT

MỘT CHẶN TRÊN CHỈ SỐ CHÍNH QUY CỦA CÁC IĐÊAN CẠNH

Cho G là một đồ thị đơn. Chúng tôi đưa ra một chặn trên cho $regI(G)$ theo số cặp cảm sinh của các cây bao trùm của đồ thị G.