A MEAN CONVERGENCE THEOREM FOR TRIANGULAR ARRAYS OF ROWWISE AND PAIRWISE m_n-DEPENDENT RANDOM VARIABLES

Le Van Thanh[1, *], Pham Nhu Y[2]

[1]*Department of Mathematics, School of Education, Vinh University, Vietnam*

[2]*Kiem Tan High School, Gia Tan 2, Thong Nhat district, Dong Nai province, Vietnam*

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****Correspondence:***
levt@vinhuni.edu.vn

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ABSTRACT

This paper establishes a mean convergence theorem for triangular arrays of rowwise and pairwise m_n-dependent random variables. Some authors studied limit theorems for sequences of pairwise m-dependent random variables where m is fixed (see, e.g., Quang and Nguyen [Applications of Mathematics, 2016] and Thanh [Bulletin of the Institute of Mathematics Academia Sinica, 2005]). In this paper, we establish a limit theorem for triangular arrays of rowwise and pairwise m_n-dependent random variables, where m_n may approach infinity as $n \to \infty$. The main theorem extends some results in the literature, including Theorem 3.1 of Chen, Bai and Sung in [Journal of Mathematical Analysis and Applications, 2014].

Keywords: Convergence in mean; Pairwise m_n-dependence, Uniform integrability in the Cesàro sense; Triangular array of random variables.

1. Introduction

In [4], Pyke and Root proved a theorem on mean convergence for sequences of independent identically distributed (i.i.d.) random variables. Their result reads as follows.

Theorem 1.1 (Pyke and Root [4]). *Let $1 \le p < 2$ and let $\{Xn, n \ge 1\}$ be a sequence of i.i.d. mean zero random variables. Then*

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n} X_i \xrightarrow{\mathcal{L}_p} 0 \ \ as\ n \to \infty$$

if and only if $\mathrm{E}|X1|^p < \infty$.

Chow [2] gave an extension of the sufficiency part of Theorem 1.1 by establishing a meanconvergence theorem under a uniform integrability condition. A special case of Chow's result reads as follows.

Theorem 1.2 (Chow [2]). *Let $1 \le p < 2$ and let $\{X_n, n \ge 1\}$ be a sequence of independent random variables such that the sequence $\{|X_n|^p, n \ge 1\}$ is uniformly integrable. Then*

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}(X_i-\mathbb{E}X_i)\xrightarrow{\mathcal{L}_p} 0 \text{ as } n\to\infty.$$

Theorem 1.2 was extended and improved in many directions. We refer to [1, 3, 7] and the references therein. A crucial tool in the proof of Theorem 1.2 and its extensions is a von Bahr–Esseen-type inequality which states that

$$\mathbb{E}\left|\sum_{i=1}^{n}X_i\right|^p \le C_p\sum_{i=1}^{n}\mathbb{E}|X_i|^p,$$

where $1\le p\le 2$, $\{X_i, 1\le i\le n\}$ is a collection of mean zero random variables satisfying a certain dependence structure, and C_p is a constant depending only on p. Chen, Bai and Sung [1, Theorem 3.1] extended Theorem 1.2 to the case where the underlying random variables are pairwise independent. This note aims to extend Theorem 1.2 to triangular arrays of rowwise and pairwise m_n-dependent mean zero random variables, thereby extending Theorem 3.1 of Chen, Bai and Sung [1].

Laws of large numbers for sequences of pairwise m-dependent random variables were studied by Quang and Nguyen [5] and Thanh [8]. Let m be a nonnegative integer. A collection $\{X_i, 1\le i\le n\}$ of random variables is said to be *pairwise m-dependent* if either $n\le m+1$ or $n>m+1$ and X_i is independent of X_j whenever $|i-j|>m$. When $m=0$, this reduces to the concept of pairwise independence. If $m'>m$, then pairwise m-dependence implies pairwise m'-dependence.

Let $\{m_n, n\ge 1\}$ be a sequence of nonnegative integers. A triangular array $\{X_{n,i}, 1\le i\le n, n\ge 1\}$ of random variables is said to be *rowwise and pairwise m_n-dependent* if for each $n\ge 1$, the n-th row $\{X_{n,i}, 1\le i\le n\}$ is pairwise m_n-dependent.

2 Main result

In this section, we will extend Theorem 1.2 by establishing a mean convergence theorem for triangular arrays of rowwise and pairwise m_n-dependent random variables. Firstly, we will need the following lemmas. The first lemma is Theorem 2.1 of Chen, Bai and Sung [1].

Lemma 2.1. *Let $1\le p\le 2$. Let $\{X_i, 1\le i\le n\}$ be a collection of pairwise independent mean zero random variables satisfying $\mathbb{E}|X_i|^p<\infty$, $1\le i\le n$. Then*

$$\mathbb{E}\left|\sum_{i=1}^{n}X_i\right|^p \le C_p\sum_{i=1}^{n}\mathbb{E}|X_i|^p,$$

where C_p is a constant depending only on p.

Remark 2.2. *In the case where $p = 1$ or $p = 2$, it is clear that we can choose $C_p = 1$.*

The next lemma is a consequence of Hölder's inequality (see, e.g., Lemma 2.4 in Rosalsky and Thanh [6]).

Lemma 2.3. *Let $p \geq 1$ and let $\{a_i, 1 \leq i \leq n\}$ be a collection of real numbers. Then*

$$\left|\sum_{i=1}^{n} a_i\right|^p \leq n^{p-1} \sum_{i=1}^{n} |a_i|^p.$$

The following lemma extends Lemma 2.1 to the case pairwise m-dependent.

Lemma 2.4. *Let m be a nonnegative integer and $1 \leq p \leq 2$. Let $\{X_i, 1 \leq i \leq n\}$ be a collection of pairwise m-dependent mean zero random variables satisfying $\mathbb{E}|X_i|^p < \infty$, $1 \leq i \leq n$. Then*

$$\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p \leq C_p (m+1)^{p-1} \sum_{i=1}^{n} \mathbb{E}|X_i|^p, \tag{2.1}$$

where C_p is a constant depending only on p. In the case where $p = 1$ or $p = 2$, we can choose $C_p = 1$.

Proof. If $n \leq m + 1$, then (2.1) follows immediately from Lemma 2.3. Suppose that $n > m + 1$. Then

$$\begin{aligned}
\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p &= \mathbb{E}\left|\sum_{k=1}^{m+1} \sum_{0 \leq i(m+1) \leq n-k} X_{i(m+1)+k}\right|^p \\
&\leq (m+1)^{p-1} \sum_{k=1}^{m+1} \mathbb{E}\left|\sum_{0 \leq i(m+1) \leq n-k} X_{i(m+1)+k}\right|^p \\
&\leq C_p (m+1)^{p-1} \sum_{k=1}^{m+1} \sum_{0 \leq i(m+1) \leq n-k} \mathbb{E}\left|X_{i(m+1)+k}\right|^p \\
&= C_p (m+1)^{p-1} \sum_{i=1}^{n} \mathbb{E}|X_i|^p,
\end{aligned}$$

where we have applied Lemma 2.3 in the first inequality, Lemma 2.1 and Remark 2.2 in the second inequality. The proof of Lemma 2.4 is completed. $\square$

Remark 2.5. Let $\{m_n, n \geq 1\}$ be a sequence of nonnegative integers and let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise and pairwise m_n-dependent mean zero random variables. Then for each $n \geq 1$, we have

$$\mathbb{E}\left|\sum_{i=1}^{n} X_{n,i}\right|^p \leq C_p(m_n+1)^{p-1}\sum_{i=1}^{n}\mathbb{E}|X_{n,i}|^p,$$

where C_p is a constant depending only on p.

The main result of this note is the following theorem. Throughout the proof of Theorem 2.6, C_p is a constant depending only on p and is not necessarily the same one in each appearance.

Theorem 2.6. *Let $1 \leq p < 2$, let $\{m_n, n \geq 1\}$ be a sequence of nonnegative integers and let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise and pairwise m_n-dependent random variables such that $\{|X_{n,i}|^p, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense, that is,*

$$\lim_{a\to\infty}\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}(|X_{n,i}|>a)\right)=0.$$

Then

$$\frac{1}{n^{1/p}(m_n+1)^{1/2}}\sum_{i=1}^{n}(X_{n,i}-\mathbb{E}X_{n,i}) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n\to\infty. \tag{2.2}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\{|X_{n,i}|^p, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense, there exists $M > 0$ such that

$$\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}(|X_{n,i}|>M)\right)<\varepsilon. \tag{2.3}$$

For $n \geq 1, 1 \leq i \leq n$, set

$$Y_{n,i} = X_{n,i}\mathbf{1}(|X_{n,i}| \leq M)$$

and

$$Z_{n,i} = X_{n,i}\mathbf{1}(|X_{n,i}| > M).$$

Then

$$\begin{aligned}\mathbb{E}\left|\sum_{i=1}^{n}(X_{n,i}-\mathbb{E}X_{n,i})\right|^p &\leq 2\left(\mathbb{E}\left|\sum_{i=1}^{n}(Y_{n,i}-\mathbb{E}Y_{n,i})\right|^p+\mathbb{E}\left|\sum_{i=1}^{n}(Z_{n,i}-\mathbb{E}Z_{n,i})\right|^p\right)\\ &:= 2(I_1+I_2).\end{aligned} \tag{2.4}$$

Applying Jensen's inequality and Lemma 2.4, we have

$$\begin{aligned} I_1 &\le \left(\mathbb{E}\left(\sum_{i=1}^{n}(Y_{n,i}-\mathbb{E}Y_{n,i})\right)^2\right)^{p/2} \\ &\le \left((m_n+1)\sum_{i=1}^{n}\mathbb{E}(Y_{n,i}-\mathbb{E}Y_{n,i})^2\right)^{p/2} \\ &\le \left((m_n+1)\sum_{i=1}^{n}\mathbb{E}Y_{n,i}^2\right)^{p/2} \\ &\le (m_n+1)^{p/2}n^{p/2}M^p. \end{aligned} \tag{2.5}$$

Applying (2.3) and Lemma 2.4 again, we have

$$\begin{aligned} I_2 &\le (m_n+1)^{p-1}C_p\sum_{i=1}^{n}\mathbb{E}|Z_{n,i}-\mathbb{E}Z_{n,i}|^p \\ &\le (m_n+1)^{p-1}C_p\sum_{i=1}^{n}\mathbb{E}|Z_{n,i}|^p \\ &\le (m_n+1)^{p/2}C_p\sum_{i=1}^{n}\mathbb{E}|Z_{n,i}|^p \\ &\le (m_n+1)^{p/2}C_p n\varepsilon. \end{aligned} \tag{2.6}$$

Combining (2.4)–(2.6) yields

$$\begin{aligned} \mathbb{E}\left|\frac{\sum_{i=1}^{n}(X_{n,i}-\mathbb{E}X_{n,i})}{n^{1/p}(m_n+1)^{1/2}}\right|^p &\le \frac{2(I_1+I_2)}{n(m_n+1)^{p/2}} \\ &\le \frac{2M^p}{n^{1-p/2}}+C_p\varepsilon. \end{aligned} \tag{2.7}$$

Since $p<2$ and $\varepsilon>0$ is arbitrary, (2.2) follows from (2.7) by letting $\varepsilon\to 0$ and then $n\to\infty$. The proof of the theorem is completed. □

Remark 2.7. If $m_n\equiv 0$, then Theorem 2.6 reduces to Theorem 3.1 of Chen, Bai and Sung [1].

We close the paper by considering a case where $m_n\to\infty$ as $n\to\infty$. In the following corollary, for $x\ge 0$, let $\lfloor x\rfloor$ denote the greatest integer that is not greater than x and let $\log x$ denote the natural logarithm of $(x+2)$.

Corollary 2.8. *Let $1\le p<2$ and let $\{X_{n,i}, 1\le i\le n, n\ge 1\}$ be a triangular array of rowwise and pairwise $\lfloor\log n\rfloor$-dependent random variables such that $\{|X_{n,i}|^p, 1\le i\le$*

$n, n \geq 1\}$ *is uniformly integrable in the Cesàro sense. Then*

$$\frac{1}{n^{1/p}\log^{1/2}(n)}\sum_{i=1}^{n}(X_{n,i} - \mathbb{E}X_{n,i}) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty.$$

Proof. Applying Theorem 2.6 for the case where $m_n \equiv \lfloor \log n \rfloor$, we immediately obtain the conclusion of the corollary. $\square$

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TÓM TẮT

MỘT SỐ ĐỊNH LÝ VỀ SỰ HỘI TỤ THEO TRUNG BÌNH CỦA MẢNG TAM GIÁC CÁC BIẾN NGẪU NHIÊN m_n-PHỤ THUỘC ĐÔI MỘT THEO HÀNG

Lê Văn Thành[1], **Phạm Như Ý**[2]

[1] *Khoa Toán, Trường Sư phạm, Trường Đại học Vinh, Việt Nam*

[2] *Trường Trung học phổ thông Kiệm Tân, Gia Tân 2, Thống Nhất, Đồng Nai, Việt Nam*

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Bài báo này thiết lập một định lý về sự hội tụ theo trung bình của mảng tam giác các biến ngẫu nhiên m_n-phụ thuộc đôi một. Một số tác giả đã nghiên cứu các định lý giới hạn cho dãy các biến ngẫu nhiên m-phụ thuộc đôi một, trong đó m cố định (xem, chẳng hạn, Quang and Nguyen [Applications of Mathematics, 2016] và Thanh [Bulletin of the Institute of Mathematics Academia Sinica, 2005]). Trong bài báo này, chúng tôi thiết lập một định lý giới hạn cho mảng tam giác các biến ngẫu nhiên m_n-phụ thuộc đôi một theo hàng, trong đó m_n có thể tiến đến ∞ khi $n \to \infty$.

Định lý chính của bài báo mở rộng một số kết quả đã công bố trước đó, trong đó có Định lý 3.1 của Chen, Bai và Sung trong [Journal of Mathematical Analysis and Applications, 2014].

Từ khóa: Sự hội tụ theo trung bình; m_n-phụ thuộc đôi một; tính khả tích đều theo nghĩa Cesàro; mảng tam giác các biến ngẫu nhiên.