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ON THE CONVERGENCE OF A COUPLING SUCCESSIVE APPROXIMATION METHOD FOR SOLVING DUFFING EQUATION

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Abstract. General Duffing equations occur in many problems of Mechanics and Dynamics. These equations include nonlinear terms of second an third order, their coefficients are finite but not small parameters. For finding analytical approximate solutions of the general Duffing equation the coupling successive approximation method (CSAM) has been proposed by the authors. In the present paper the convergence of mentioned method is proven and a condition relating coefficients of Duffing equation to provide the convergence procedure is formulated Emphasize that the assumption of small parameters is not used in the proving. Some examples are presented to illustrate the proposed method, particularly exact solutions of some problems are compared with analytical approximate ones found by CSAM

Keywords. General Duffing equation, coupling successive approximation method, convergence, complex valued solution, chaotic solution.

1. INTRODUCTION

General Duffing equations appear in formulating and solving many problems of Mechanics and Dynamics, for example [1–7]. Different methods for finding analytical approximate solutions to nonlinear differential equations have been proposed, but the convergence are not to be established for all these methods

For the successive approximation method to nonlinear differential equation of first order [8] (p. 270) and linear differential equation of second order with functional coefficients [9] (p. 317) the convergence condition was indicated, but for nonlinear ones it is still open.

Elastic solution method [10] applying to an elastic-plastic problem leads to solve successive elastic problems. The convergence of the method was proven [11].

Averaging methods are analytical approximate methods for that the convergence was proven using assumption of small parameters [12].

Homotopy analysis method (HAM) [13] is based on the expansion of solution into Taylor series, the convergence was proven by comparison of HAM solution with solution obtained by numerical method.

The convergence of energy balance method (EBM) [14], variational approach method (VAM) [15], parameter expansion method (PEM) [16] was proven by comparison of these solutions with exact solution in particular cases.

The convergence of coupling successive approximation method (CSAM) will be proven analytically based on two propositions.

The focus of interests in this paper includes:

- To prove the convergence of CSAM, not using the assumption of small parameters in solving and proving procedure and to indicate the condition providing convergence process.

- To solve some particular problems by CSAM and to investigate the characteristics of solutions. Particularly exact solutions found for some problems are compared with analytical approximate solutions by CSAM, from that one can estimate the accuracy of CASM.

2. THE ALGORITHM TO TRANSFORM THE INITIAL EQUATION TO THE RESULTING EQUATION

Consider a general Duffing equation as

$$\ddot{x} + 2\nu\dot{x} + \lambda x^3 + 2qx^2 + kx = p\cos\omega t. \tag{1}$$

Based on [1] a transformation is considered as

$$x = -\frac{2}{3\lambda}\left[q + 4\nu\sqrt{-\frac{\lambda}{8}}\right] + \frac{1}{2\sqrt{\frac{-\lambda}{8}}}\frac{\dot{\xi}}{\xi}. \tag{2}$$

The resulting equation is obtained

$$\ddot{\xi} + 2\nu\dot{\xi} - \frac{1}{2}K\xi = f(\xi, t), \tag{3}$$

where

$$f(\xi,t) = 2\sqrt{-\frac{\lambda}{8}}\xi^3\left[D_2 + \int_0^t (\sigma + p\cos\omega t)\frac{1}{\xi^2}dt\right], \tag{4}$$

$$K = k - \frac{4}{3}\frac{q^2}{\lambda} - \frac{8}{3}\nu^2, \tag{5}$$

$$\sigma = -\frac{2}{3\lambda}\left(q + 4\nu\sqrt{-\frac{\lambda}{8}}\right)\left[\frac{8}{9\lambda}\left(q + 4\nu\sqrt{-\frac{\lambda}{8}}\right)\left(q - 2\nu\sqrt{-\frac{\lambda}{8}}\right) - k\right], \tag{6}$$

D_2 is an integral constant. The formula (2) and Eq. (3) are the transformation and resulting equation that the present paper is looking for. In order to formulate a successive approximation method, the right hand side of Eq. (3) is rewritten as

$$f(\xi,t) = 2\sqrt{\frac{-\lambda}{8}}\eta(\xi,t)\xi, \tag{7}$$

where

$$\eta(\xi, t) = \xi^2 \left[D_2 + \int_0^t (\sigma + p\cos\omega t) \frac{1}{\xi^2} dt \right]. \tag{8}$$

3. EQUATION SOLVING BY THE COUPLING SUCCESSIVE APPROXIMATION METHOD

An analytical approximate solution to Eq. (3) by the coupling successive approximation method is carried out by continuous loops of iteration. Each loop contains continuously iterative steps.

3.1. Loops of iteration [1]

In the loop "0"th, we solve the linear differential equation (3) without the right hand side to find the solution $\xi_0(t)$. In the first loop, substituting $\xi(t) = \xi_0(t)$ in the right hand side of Eq. (3) and solving the obtained linear differential equation we find $\xi_1(t)$ and so on. In loop $n-1^{\text{th}}$, the value $\xi_{n-1}(t)$ is found. The function $\eta(\xi_{n-1}, t)$ is computed by the formula (8)

$$\eta(\xi_{n-1}, t) = (\xi_{n-1})^2 \left[D_2 + \int_0^t (\sigma + p\cos\omega t) \frac{1}{(\xi_{n-1})^2} dt \right]. \tag{9}$$

The iteration scheme of successive approximation method is introduced as follows

$$\ddot{\xi}_n + 2\nu\dot{\xi}_n - \frac{1}{2}K\xi_n = 2\sqrt{-\frac{\lambda}{8}}\eta(\xi_{n-1}, t)\xi_{n-1}. \qquad n = 1, 2, 3\ldots \tag{10}$$

By solving Eq. (10), where the right hand side is a known function, the analytical approximate solution in n^{th} loop of iteration is obtained

$$\xi_n = \frac{\sqrt{\frac{-\lambda}{8}}}{\theta} y_{n-1} - \frac{\sqrt{\frac{-\lambda}{8}}}{\theta} z_{n-1} + D_3 e^{-(\nu-\theta)t} - D_4 e^{-(\nu+\theta)t}, \tag{11}$$

where

$$y_{n-1}(t) = e^{-(\nu-\theta)t} \left[\int_0^t \eta(\xi_{n-1}, t)\xi_{n-1} e^{(\nu-\theta)t} dt \right], \tag{12}$$

$$z_{n-1}(t) = e^{-(\nu+\theta)t} \int_0^t \eta(\xi_{n-1}, t)\, \xi_{n-1} e^{(\nu+\theta)t} dt, \tag{13}$$

$$\theta = \left[\frac{1}{2}\left(k - \frac{4}{3}\frac{q^2}{\lambda} - \frac{2}{3}\nu^2 \right) \right]^{\frac{1}{2}} \tag{14}$$

Examining Eq. (11) we can predict some characteristics of solution.

If $k - \frac{4}{3}\frac{q^2}{\lambda} - \frac{2}{3}\nu^2 > 0$ then θ is a real number, the solution describes an oscillation depending on the excitation frequency ω.

If $k - \frac{4}{3}\frac{q^2}{\lambda} - \frac{2}{3}\nu^2 < 0$ then θ is an imaginary number, i.e

$$\theta = i\varphi \quad \text{with} \quad \varphi = \left[\frac{1}{2}\left(\frac{2}{3}\nu^2 + \frac{4}{3}\frac{q^2}{\lambda} - k\right)\right]^{1/2}$$

where φ plays the role of the new frequency of a nonlinear vibration. The solution (11) describes a complex oscillation with many frequencies: excitation frequency ω, vibration frequency φ and combined frequency of ω and φ, so that the chaotic characteristics of solution may be predicted.

Each function in the sequence $\xi_0(t), \xi_1(t), \ldots, \xi_{n-1}(t), \xi_n(t)$ can be determined from the one immediately proceeding it by solving the respective linear differential equation (10).

The process is stopped when the condition $\max_n \|\xi_n(t) - \xi_{n-1}(t)\| < \varepsilon$ is achieved, where ε is a small positive number as required. But the convergence proving of this process is very complicated. Thus, a coupling successive approximation method based on Eqs. (3) and (7) must be developed with the iterative steps as follows: in each loop of iteration, continuously iterative steps are carried out.

3.2. Iterative steps in each loop [1]

In the loop n^{th}, when the iterative step m^{th} is carried out, the value $\eta(\xi_{n-1}, t)$ is known. This value is taken at the end of the previous loop (loop $(n-1^{\text{th}})$). At this point, the iteration scheme of the coupling successive approximation method for the loop n^{th} and the iterative step m^{th} is expressed as

$$\ddot{\xi}_{n,m} + 2\nu\dot{\xi}_{n,m} - \frac{1}{2}K\xi_{n,m} = 2\sqrt{\frac{-\lambda}{8}}\eta(\xi_{n-1}, t)\,\xi_{n,m-1}, n = 1,2,3\ldots, m = 1,2,3\ldots \tag{15}$$

where n denotes the number of loop and m - the number of iterative step.

The approximate solution $\xi_{n-1}(t)$ in the last loop $n - 1^{\text{th}}$ is taken as an initial approximation at the iterative step "0"$^{\text{th}}$ of the loop n^{th}, denoted as $\xi_{n,0}(t)$. Thus, that requires

$$\xi_{n-1}(t) = \xi_{n,0}(t).$$

Solving Eq. (15), where the right hand side is a known function, we have

$$\xi_{n,m} = \frac{\sqrt{\frac{-\lambda}{8}}}{\theta}y_{n,m-1} - \frac{\sqrt{\frac{-\lambda}{8}}}{\theta}z_{n,m-1} + D_3e^{-(\nu-\theta)t} - D_4e^{-(\nu+\theta)t}, \tag{16}$$

where

$$\begin{aligned} y_{n,m-1}(t) &= e^{-(\nu-\theta)t}\left[\int_0^t \eta(\xi_{n,0}, t)\xi_{n,m-1}e^{(\nu-\theta)t}dt\right], \\ z_{n,m-1}(t) &= e^{-(\nu+\theta)t}\int_0^t \eta(\xi_{n,0}, t)\,\xi_{n,m-1}e^{(\nu+\theta)t}dt. \end{aligned} \tag{17}$$

From which

$$\begin{aligned}\dot{\xi}_{n,m} &= -(\nu-\theta)\frac{\sqrt{\frac{-\lambda}{8}}}{\theta}y_{n,m-1}+(\nu+\theta)\frac{\sqrt{\frac{-\lambda}{8}}}{\theta}z_{n,m-1}-(\nu-\theta)D_3e^{-(\nu-\theta)t}\\ &\quad+(\nu+\theta)D_4e^{-(\nu+\theta)t},\\ \ddot{\xi}_{n,m} &= (\nu-\theta)^2\frac{\sqrt{\frac{-\lambda}{8}}}{\theta}y_{n,m-1}-(\nu+\theta)^2\frac{\sqrt{\frac{-\lambda}{8}}}{\theta}z_{n,m-1}+(\nu-\theta)^2D_3e^{-(\nu-\theta)t}\\ &\quad-(\nu+\theta)^2D_4e^{-(\nu+\theta)t}+2\sqrt{\frac{-\lambda}{8}}\eta(\xi_{n,0},t)\,\xi_{n,m-1}.\end{aligned} \tag{18}$$

Remarks: If each loop is carried out with only one step, the coupling successive method will return to the single successive method as mentioned in Section 3.1.

4. CONVERGENCE OF THE COUPLING SUCCESSIVE APPROXIMATION METHOD

In order to prove the convergence of the method, it need to prove the following two propositions:

Proposition 1

If $\eta(\xi,t)$ obtained from Eq. (8) and $|\xi(t)|\le M$, then $|\eta(\xi,t)|\le N$.

Proof:

In order to prove this proposition the method of contradiction is used:
Differentiating with respect to t both sides of Eq. (8), we have

$$\dot{\eta}=2\xi\dot{\xi}\left[D_2+\int_0^t(\sigma+p\cos\omega t)\frac{1}{\xi^2}dt\right]+\sigma+p\cos\omega t.$$

Using Eq. (8) this equation can be rewritten as

$$\frac{\dot{\eta}}{\eta}-2\frac{\dot{\xi}}{\xi}=\frac{\sigma+p\cos\omega t}{\eta}.$$

Integrating both sides with respect to t yields

$$\frac{\eta(t)}{\xi^2(t)}=C\exp\left[\int_0^t\frac{\sigma+p\cos t}{\eta(t)}dt\right],$$

where the integral coefficient C is a positive number, from which

$$\left|\frac{\eta(t)}{\xi^2(t)}\right|\le C\exp\left[\int_0^t\frac{|\sigma+p\cos t|}{|\eta(t)|}dt\right].$$

The inequality obtained can be rewritten as

$$|\eta(t)|\exp\left[-\int_0^t\frac{|\sigma+p\cos t|}{|\eta(t)|}dt\right]\le C\left|\xi^2(t)\right|. \tag{19}$$

From the inequality (19) it can be concluded that if $|\xi(t)| < M$ then $|\eta(\xi, t)| < N$, because if $|\eta(\xi, t)| \to +\infty$, the left hand side of the inequality leads to the infinity, which is contradictory with the assumption that $\xi(t)$ is bounded.

Proposition 2

If $|\eta(\xi, t)| < N$, then the sequence of functions obtained in iterative steps of each loop will converge and converge on function $\xi(t)$ with $|\xi(t)| < M$.

Proof: The recurrence method to prove this proposition is used as following. First, prove the convergence of the method in the first loop ($n = 1$).

The 0-th approximation can be selected: $\xi_0 = D_3 e^{-(\nu-\theta)t}$ with $|\xi_0| \le |D_3| e^{-\nu t}$ or

$$\xi_0 = D_4 e^{-(\nu+\theta)t} \quad \text{with} \quad |\xi_0| \le |D_4| e^{-\nu t}.$$

where $\xi_0(t)$ is the solution of the homogenous equation (3), which is taken as the initial approximation solution of the first loop, i.e. $\xi_0 = \xi_{1,0}$ (it is considered as an approximation in the "0"$^{\text{th}}$ step of the first loop).

From Eq. (16), with $n = 1, m = 1$ we have

$$\xi_{1,1} = \frac{\sqrt{\frac{-\lambda}{8}}}{\theta} y_{1,0}(t) - \frac{\sqrt{\frac{-\lambda}{8}}}{\theta} z_{1,0}(t) + \xi_{1,0} - D_4 e^{-(\nu+\theta)t}. \tag{20}$$

Similarly, with $n = 1, m = 2$ we obtain

$$\xi_{1,2} = \frac{\sqrt{\frac{-\lambda}{8}}}{\theta} y_{1,1}(t) - \frac{\sqrt{\frac{-\lambda}{8}}}{\theta} z_{1,1}(t) + \xi_{1,0} - D_4 e^{-(\nu+\theta)t}, \tag{21}$$

where

$$y_{1,0}(t) = e^{-(\nu-\theta)t} \int_0^t \eta(\xi_{1,0}, t)\, \xi_{1,0} e^{(\nu-\theta)t} dt, \tag{22}$$

$$y_{1,1}(t) = e^{-(\nu-\theta)t} \int_0^t \eta(\xi_{1,0}, t)\, \xi_{1,1} e^{(\nu-\theta)t} dt, \tag{23}$$

$$z_{1,0}(t) = e^{-(\nu+\theta)t} \int_0^t \eta(\xi_{1,0}, t)\, \xi_{1,0} e^{(\nu+\theta)t} dt, \tag{24}$$

$$z_{1,1}(t) = e^{-(\nu+\theta)t} \int_0^t \eta(\xi_{1,0}, t)\, \xi_{1,1} e^{(\nu+\theta)t} dt. \tag{25}$$

Because $|\xi_{1,0}|$ is bounded, according to Proposition 1 $|\eta(\xi_{1,0}, t)| < N$, two cases would exist:

a) Case 1: $\theta = i\varphi$, φ is real.

To facilitate the approximation process, the following equivalence relations are used

$$\left|e^{\pm i\varphi t}\right| = |\cos(\varphi t) \pm i \sin(\varphi t)| = \cos^2(\varphi t) + \sin^2(\varphi t) = 1.$$

Eqs. (20), (22) and (24) give

$$|\xi_{1,1} - \xi_{1,0}| \le \left(|D_3| \frac{\rho N t}{\varphi} + |D_4|\right) e^{-\nu t}, \quad \rho = 2\left|\sqrt{\frac{-\lambda}{8}}\right|. \tag{26}$$

Subtracting each side of Eqs. (20) and (21) respectively, taking into account Eqs. (23) and (25), we have

$$\begin{aligned}\xi_{1,2}-\xi_{1,1}=&\frac{\sqrt{\frac{-\lambda}{8}}}{\theta}e^{-(\nu-\theta)t}\int_0^t\eta\left(\xi_{1,0},t\right)\left(\xi_{1,1}-\xi_{1,0}\right)e^{(\nu-\theta)t}dt\\&-\frac{\sqrt{\frac{-\lambda}{8}}}{\theta}e^{-(\nu+\theta)t}\int_0^t\eta\left(\xi_{1,0},t\right)\left(\xi_{1,1}-\xi_{1,0}\right)e^{(\nu+\theta)t}dt.\end{aligned}\tag{27}$$

From Eqs. (26) and (27) one can evaluate

$$|\xi_{1,2}-\xi_{1,1}|\le\frac{\rho N}{\varphi}\left(|D_3|\frac{\rho Nt^2}{\varphi 2!}+|D_4|\,t\right)e^{-\nu t}\tag{28}$$

Repeating the next steps in the first loop based on Eq. (28), leads to

$$|\xi_{1,m}-\xi_{1,m-1}|\le\left(\frac{\rho N}{\varphi}\right)^{m-1}\left(|D_3|\frac{\rho N}{\varphi}\frac{t^m}{m!}+|D_4|\frac{\rho N}{\varphi}\frac{t^{m-1}}{(m-1)!}\right)e^{-\nu t}\tag{29}$$

$$|\xi_{1,m+1}-\xi_{1,m}|\le\left(\frac{\rho N}{\varphi}\right)^{m}\left(|D_3|\frac{\rho N}{\varphi}\frac{t^{m+1}}{(m+1)!}+|D_4|\frac{\rho N}{\varphi}\frac{t^{m}}{m!}\right)e^{-\nu t}.\tag{30}$$

The terms of the series are directly obtained

$$\xi_{1,0}+(\xi_{1,1}-\xi_{1,0})+(\xi_{1,2}-\xi_{1,1})+\ldots+(\xi_{1,m+1}-\xi_{1,m})+\ldots\tag{31}$$

As can be seen that with $t<R$ each term of series (31) has a module which is smaller than a positive number. These positive numbers form a numerical convergent series, according to J. d'Alembert criterion

$$\frac{\left(\frac{\rho N}{\varphi}\right)^m\left[|D_3|\frac{\rho N}{\varphi}\frac{t^{m+1}}{(m+1)!}+|D_4|\frac{t^m}{m!}\right]}{\left(\frac{\rho N}{\varphi}\right)^{m-1}\left[|D_3|\frac{\rho N}{\varphi}\frac{t^m}{m!}+|D_4|\frac{t^{m-1}}{(m-1)!}\right]}=\frac{\rho Nt}{\varphi m}\frac{|D_3|\frac{\rho N}{\varphi}\frac{t}{m+1}+|D_4|}{|D_3|\frac{\rho N}{\varphi}\frac{t}{m}+|D_4|}\to 0,$$

when $m\to+\infty$.

That means the series (B1) is absolutely convergent when $t<R$. The sum of the first $m+1$ terms of the series is $\xi_{1,m+1}$. Thus, $\xi_{1,m+1}$ converges on the function $\xi_1(t)$ with $|\xi_1(t)|<M$ when $t<R$, and Proposition 2 is proved in the first loop.

$\xi_1(t)$ is an approximate solution obtained when the first loop ended, which is then used as the initial approximation in the second loop, i.e. $\xi_1=\xi_{2,0}$ (which is considered as an approximation in step "0"th of the second loop).

According to Proposition 2, $\xi_{2,0}$ obtained in the first loop is bounded $|\xi_{2,0}(t)|<M$. Thus, according to Proposition 1, $\eta(\xi_{2,0},t)$ from Eq. (8) with $n=2$ is also bounded $|\eta(\xi_{2,0},t)|<N$. When $\eta(\xi_{2,0},t)$ is bounded, according to Proposition 2, $\xi_{3,0}$ obtained when the second loop ended is also bounded. Thus, proposition 2 is proved for the second loop, and similarly, for the n^{th} loop.

b) Case 2: θ is real, $\nu-\theta>0$.

In this case, in order to facilitate the approximation process, the following inequalities are used

$$\left|D_3 e^{-(\nu-\theta)t}\right| \leq |D_3|, \left|D_4 e^{-(\nu+\theta)t}\right| \leq |D_4|, 1-e^{-(\nu-\theta)t} \leq 1, 1-e^{-(\nu+\theta)t} \leq 1. \tag{32}$$

From Eqs. (20), (22) and (24) one can assess

$$|\xi_{1,1}-\xi_{1,0}| \leq \frac{\rho}{2\theta}\frac{N|D_3|}{\nu-\theta}+\frac{\rho}{2\theta}\frac{N|D_3|}{\nu+\theta}+|D_4| < \frac{\rho}{\theta}\frac{N|D_3|}{\nu-\theta}+|D_4|. \tag{33}$$

Based on Eqs. (27) and (33), it leads to

$$|\xi_{1,2}-\xi_{1,1}| < \frac{\rho N}{\theta(\nu-\theta)}\left[\frac{\rho N}{\theta(\nu-\theta)}|D_3|+|D_4|\right]. \tag{34}$$

Conducting similar steps with reference to (34), we obtain the following evaluation

$$|\xi_{1,m}-\xi_{1,m-1}| < \left(\frac{\rho N}{\theta(\nu-\theta)}\right)^{m-1}\left(\frac{\rho N}{\theta(\nu-\theta)}|D_3|+|D_4|\right), \tag{35}$$

$$|\xi_{1,m+1}-\xi_{1,m}| < \left(\frac{\rho N}{\theta(\nu-\theta)}\right)^{m}\left(\frac{\rho N}{\theta(\nu-\theta)}|D_3|+|D_4|\right). \tag{36}$$

From Eqs. (33)-(36), it can be directly inferred that the coefficients of the series (31) have module smaller than positive numbers. These positive numbers form a convergent series with the condition as following

$$\frac{\left(\frac{\rho N}{\theta(\nu-\theta)}\right)^{m}\left(\frac{\rho N}{\theta(\nu-\theta)}|D_3|+|D_4|\right)}{\left(\frac{\rho N}{\theta(\nu-\theta)}\right)^{m-1}\left(\frac{\rho N}{\theta(\nu-\theta)}|D_3|+|D_4|\right)} = \frac{\rho N}{\theta(\nu-\theta)} < 1. \tag{37}$$

The condition (37) is satisfied, meaning that the absolute convergence of the series (31) according to d'Alembert criterion in Proposition 2 with θ-real, $\nu-\theta>0$ is proved for the first loop. Similarly, it can be proved for the second loop and the n^{th} loop.

When θ is real, $\nu-\theta<0$, the convergence of the method has not been proven yet.

Remarks:

Through proving the convergence of the coupling successive approximation method, the following conditions for convergence are acknowledged

$$k-\frac{4}{3}\frac{q^2}{\lambda}-\frac{2}{3}\nu^2<0 \quad \text{or} \quad k-\frac{4}{3}\frac{q^2}{\lambda}-\frac{2}{3}\nu^2>0, \nu>\theta, \frac{\rho N}{\theta(\nu-\theta)}<1,$$

the condition of small parameters is not necessary

Particular case: To simplify process for seeking an analytic approximated solution, we can use a 'roughly' single successive method as follows.

Finding an approximate solution in the step n^{th} can be based on the equation

$$\ddot{\xi}_n+2\nu\dot{\xi}_n-\frac{1}{2}K\xi_n=2\sqrt{\frac{-\lambda}{8}}\eta(\xi_0,t)\xi_{n-1},$$

where ξ_0 is a solution to the linear differential equation (3) without the right hand side. Thus, using Propositions 1 and 2, the convergence of the 'roughly' single successive approximation method can be proved.

5. APPLICATIONS AND ASSESSMENT OF SOLUTION PROPERTIES

5.1. Exact solution

The proposed method can be used to find exact solutions in some particular cases:

a. Case 1

Consider an initial equation (1)

$$\ddot{x} + \lambda x^3 + 2qx^2 + kx = 0, \tag{38}$$

where: $\nu = 0, p = 0$.

Based on Eq. (2), the transformation can be written as

$$x = -\frac{2q}{3\lambda} + \frac{1}{2\sqrt{\frac{-\lambda}{8}}}\frac{\dot{\xi}}{\xi}. \tag{39}$$

From Eq. (3), with $\sigma = 0, p = 0$, the resulting equation now can be written as

$$\ddot{\xi} - \frac{1}{2}K\xi = 2C_2\xi^3,$$

where, $C_2 = \sqrt{\frac{-\lambda}{8}}D_2$ is an arbitrary constant.

$$\sigma = 0, \text{ i.e. } -\frac{2q}{3\lambda}\left(\frac{8a^2}{9\lambda} - k\right) = 0,$$

from that

$$k = \frac{8q^2}{9\lambda}.$$

According to Eq. (5), $K = k - \dfrac{4q^2}{3\lambda} = -\dfrac{4q^2}{9\lambda}$.

The resulting equation gives an exact solution.

$$\xi = \beta cn\left[(-at + \phi), k_1\right], \tag{40}$$

in which cn is an elliptic function,

$$\beta^2 = -\frac{K}{4C_2} + \sqrt{\frac{K^2}{16C_2^2} - \frac{C_1}{C_2}}, \quad \alpha^2 = \frac{K}{4C_2} + \sqrt{\frac{K^2}{16C_2^2} - \frac{C_1}{C_2}},$$

$$a = \sqrt{-2\sqrt{\frac{K^2}{16C_2^2} - \frac{C_1}{C_2}}}, \qquad k_1^2 = \frac{\beta^2}{\alpha^2 + \beta^2},$$

where: ϕ, C_1, C_2 - integral constants and k_1 is a modulus of elliptic function.

Substituting Eq. (40) into Eq. (39) yields

$$x = -\frac{2q}{3\lambda} + \frac{\sqrt{-2\sqrt{\frac{K^2}{16} - C_1C_2}dn\left[(at+\phi), k_1\right] sn\left[(at+\phi), k_1\right]}}{cn\left[(at+\phi), k_1\right]}, \tag{41}$$

where dn, sn are elliptic functions.

Because of complexity of the exact solution (41) illustrated in elliptic function we consider the particular solution (41) where $\lambda = 0.24, q = 0.66, \phi = 2, C_1 = -0.125, C_2 = 2$. This solution corresponds to initial conditions

$$x_0 = x(t)|_{t=0} = 4.76919, \quad \dot{x}_0 = \dot{x}(t)|_{t=0} = -14.1262i.$$

Solving the initial equation (38) with the same set of parameters:, $\lambda = 0.24$, $q = 0.66$, $k = \frac{8q^2}{9\lambda}, dx_0 = 4.76919$, $\dot{x}_0 = -14.1262i$ by the CSAM, and comparing the obtained corresponding results with the exact solutions (38) as demonstrated in Figs. 1a, 1b and 1c. one can see that a very good agreement is obtained.

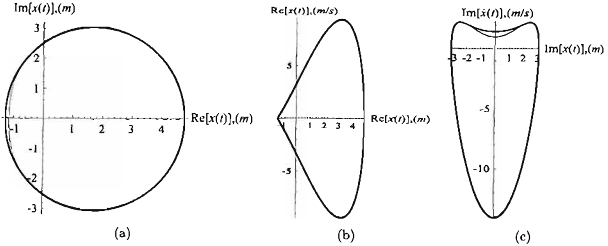

Fig. 1. Comparisons of exact solution with solution (41) at the first approximation for CSAM, continuous line-exact solution, dashed line - CSAM solution

b. Case 2

Consider an initial equation (1) in the form

$$\ddot{x} + 2\nu\dot{x} + \lambda x^3 + kx = 0. \tag{42}$$

where $q = 0, p = 0$.

In this case, according to Eq. (2) the transformation can be written as

$$x = \frac{1}{2\sqrt{\frac{-\lambda}{8}}}\left(\frac{2\nu}{3} + \frac{\dot{\xi}}{\xi}\right). \tag{43}$$

Undetermined constant is chosen as $D_2 = 4\sqrt{\frac{-\lambda}{8}}$ and parameters of the initial equation are related as $k = \frac{8\nu^2}{9}$ such that $\sigma = 0$ and the solving Eq. (3) reduces to

$$\ddot{\xi} + 2\nu\dot{\xi} + \lambda\xi^3 + k\xi = 0. \tag{44}$$

Further one can see that according to Eq. (5), $k = -\frac{1}{2}K$.

It can be observed that when $q = 0$, $p = 0$, Eq. (44) becomes Eq. (42) with the substitution $\xi = x$, the transformation (43) now becomes

$$x = \frac{1}{2\sqrt{-\frac{\lambda}{8}}}\left(\frac{2\nu}{3} + \frac{\dot{x}}{x}\right). \tag{45}$$

After some calculations we can see that the transformation (45) reduces Eq. (42) to itself. From Eq. (43) it can be inferred that

$$\xi = e^{\frac{-2\nu}{3}t}\exp\left(2\sqrt{-\frac{\lambda}{8}}\int_0^t x\mathrm{d}t\right). \tag{46}$$

Putting the new unknown

$$Z = \exp\left(2\sqrt{-\frac{\lambda}{8}}\int_0^t x\mathrm{d}t\right), \tag{47}$$

and the new variable $\tau = e^{-\frac{2\nu}{3}t}$, such that $\xi = \tau Z$ then establishing some calculations, we transform Eq. (44) into the following equation

$$Z'' + \lambda\left(\frac{3}{2\nu}\right)^2 Z^3 = 0, \tag{48}$$

where denote

$$Z' = \frac{dZ}{d\tau}, \quad Z'' = \frac{d^2Z}{d\tau^2}.$$

The solution to Eq. (48) is

$$Z = \frac{2\nu}{3\sqrt{\lambda}}c_0\sqrt{2}cn\left[\left(-c_0\sqrt{2}\tau + \phi\right), \frac{1}{2}\right]. \tag{49}$$

Rewriting the above in terms of ξ, t we have

$$\xi = \frac{2\nu}{3\sqrt{\lambda}}c_0\sqrt{2}e^{\frac{-2\nu}{3}t}cn\left[\left(-c_0\sqrt{2}e^{\frac{-2\nu}{3}t} + \phi\right), \frac{1}{2}\right], \tag{50}$$

in which: c_0, ϕ -integral constants.

The solution (50) of Eq. (42) with condition $k = \frac{8}{9}\nu^2$ can be found by others methods, such as the Lie symmetry method [17], the elliptic function method [18] and Painlevé method [19]. The method presented in this paper can be named as substitution method. This method transforms the initial equation with the condition $k = \frac{8}{9}\nu^2$ to itself. Therefore, it can be inferred that the initial equation has an infinite number of solutions.

When a solution is known, other solutions can be found based on the transformation function. From Eq. (43) it can be inferred that

$$x = \frac{1}{2\sqrt{\frac{-\lambda}{8}}}\left(\frac{2\nu}{3} - \frac{\frac{2\nu}{3}\left(\tau Z + \tau^2 Z'\right)}{\tau Z}\right) = \frac{-\nu}{3\sqrt{\frac{-\lambda}{8}}}\tau\frac{Z'}{Z}.$$

Substituting Eq. (49) into the equation and rewriting it in terms of t leads to

$$x = \frac{-4\nu}{3\sqrt{-\lambda}}c_0 e^{\frac{-2\nu}{3}t}\frac{sn\left[\left(-c_0\sqrt{2}e^{\frac{-2\nu}{3}t}+\phi\right),\frac{1}{2}\right] dn\left[\left(-c_0\sqrt{2}e^{\frac{-2\nu}{3}t}+\phi\right),\frac{1}{2}\right]}{cn\left[\left(-c_0\sqrt{2}e^{\frac{-2\nu}{3}t}+\phi\right),\frac{1}{2}\right]}. \tag{51}$$

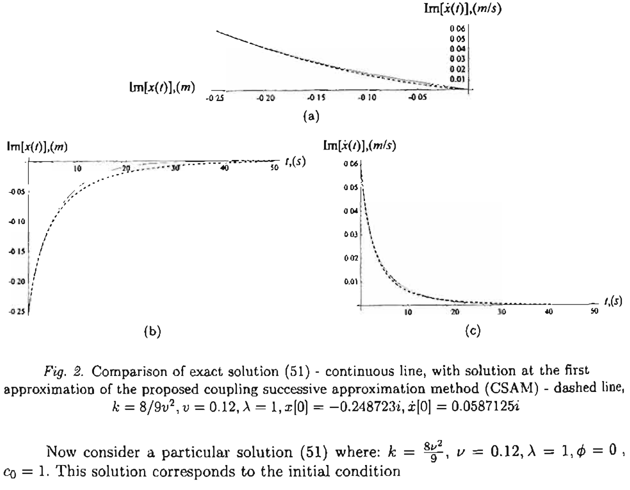

Fig. 2. Comparison of exact solution (51) - continuous line, with solution at the first approximation of the proposed coupling successive approximation method (CSAM) - dashed line, $k = 8/9v^2, v = 0.12, \lambda = 1, x[0] = -0.248723i, \dot{x}[0] = 0.0587125i$

Now consider a particular solution (51) where: $k = \frac{8\nu^2}{9}$, $\nu = 0.12, \lambda = 1, \phi = 0$, $c_0 = 1$. This solution corresponds to the initial condition

$$x_0 = x(t)|_{t=0} = -0.248723i, \quad \dot{x}_0 = \dot{x}(t)|_{t=0} = 0.0597125i.$$

Solving the initial equation (42) with the same set of parameters:, $k = \frac{8\nu^2}{9}$, $\nu = 0.12$, $\lambda = 1, x_0 = -0.248723i$, $\dot{x}_0 = 0.0597125i$ by the CSAM, and comparing the obtained results with the exact solutions (51) as illustrated in Figs. 2a, 2b and 2c, we can see that a good agreement is obtained.

5.2. Complex valued solutions

Complex-valued solutions have two components, the real and imaginary, $Re\,[x\,(t)]$, $Im\,[x\,(t)]$. Differentiated complex-valued solution with respect to time also has two components, the real and imaginary, $Re[\dot{x}(t)], Im[\dot{x}(t)]$. From Eq. (1) and the equivalent Eq. (3), the initial integral including four components mentioned above is founded. Therefore only three components are independent. These three components form a phase space, which is different from a phase plane in the case of real valued solution [20].

Consider Eq. (1) with the following set of parameters

$k = 0.6, q = 0.64, \lambda = 1.0, \nu = 0.64, \omega = 0.32, p = 2.5, x_0 = -0.4, \dot{x}_0 = -2.0.$

In this case $\lambda > 0$, from Eq. (14) θ can be evaluated as $\theta = 0.402244$, i.e. θ is real. The results obtained by CSAM are illustrated in Figs. 3-6.

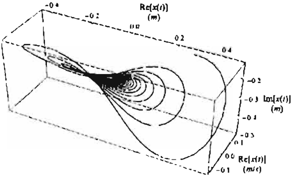

Fig. 3. Phase space with $t(30, 1000)$, based on the results at the first approximation

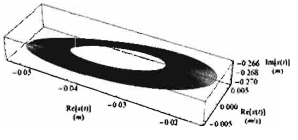

Fig. 4. Phase space with $t(1000, 2000)$, based on the results at the first approximation

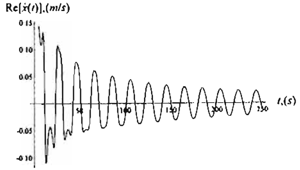

Fig. 5. The real component of solution $\dot{x}\,(t)$, based on the results at the first approximation

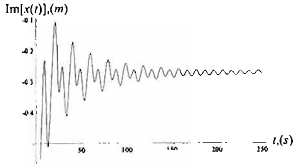

Fig. 6. The imaginary component of solution $x(t)$, based on the results at the first approximation

Remarks:

- Phase curves in Fig. 3 intersect, but in Fig. 4 they do not intersect. That means at period $t(30, 400)$ the unstable motion occurs; at period $t > 400$ the motion becomes toward a periodic one (Figs. 5 and 6).

- Fig. 4 shows that phase curves at $t > 400$ do not intersect. They are intertwined and form a closed ring.
- In this example $\theta = 0.402244, \lambda = 1$, the solution is a complex-valued one and has properties of a stable nonlinear motion.

5.3. Chaotic solution

As can be seen that the indication of the chaotic solution to the Duffing equation is shown by the factor θ (see Eq. (14)), when $\theta = i\varphi$, φ is real number.

Consider Eq. (1) with given parameters as follows:

$k = 0.0, q = 0.0, \lambda = 1.0, \nu = 0.02248, \omega = 0.44964, p = 1.0, x_0 = -0.4, \dot{x}_0 = -1.$

In this case θ can be evaluated as $\theta = 0.0129788i$, the results obtained by CSAM are presented in Figs. 7-10.

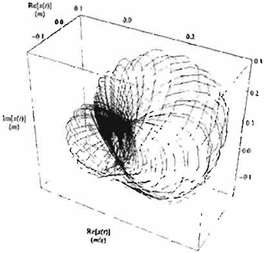

Fig. 7. Phase space with $t(150, 2100)$, based on the results at the first approximation

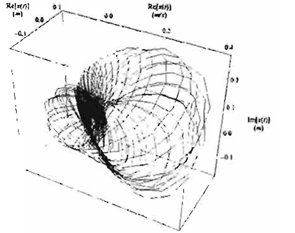

Fig. 8. Phase space with $t(2000, 4000)$, based on the results at the first approximation

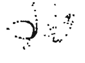

Fig. 9. Poincaré section of the phase space in Fig. 7 with $Im[x(t)] = 0$

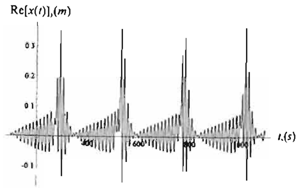

Fig. 10. The real component of solution $x(t)$, based on the results at the first approximation

Remarks:

The coefficient $\lambda = 1.0$ and the coefficient of exciting forces $p = 1.0$ have limited values, meanwhile the coefficient of the linear term $k = 0$. Thus, it is not suitable to use the assumption of small parameters in solving this problem.

In this example, $\theta = 0.0129788i$, the solution is complex-valued and chaotic one.

The curves in the phase space (Fig. 7 and 8) are rough, creased, intersecting and intertwined. The space phase has been built for $t(150, 2100)$, and for $t(2000, 4000)$, and it is possible to build a space phase for larger value t. From that, the attracting set can be built as the limit of the phase space when $t \to +\infty$.

The phase curves are sensitive with the initial condition. When the initial condition changes a little, the corresponding phase curves change a lot.

The curves of the real-valued component of solution $x(t)$ cluster together (Fig. 10). They do not repeat each other, but they have a similar structure. The clusters are thus considered sustainable.

Poincaré section (Fig. 9) consists of a set of points. Thus, the chaotic property of the solution in this example is proved.

6. CONCLUSION

Findings of the paper are summarized as follows:

1. The convergence of the coupling successive approximation method (CSAM) is proved for Eq. (1) without using the assumption of small parameters.

2. Condition of convergence is obtained as follows

$$k - \frac{4}{3}\frac{q^2}{\lambda} - \frac{2}{3}\nu^2 < 0, \quad \text{or} \quad k - \frac{4}{3}\frac{q^2}{\lambda} - \frac{2}{3}\nu^2 > 0 \quad \text{and} \quad \nu - \theta > 0,$$

where θ is denoted by (14).

3. The proposed algorithm is applied to some examples to verify the method and assess the properties of solutions.

4. Using procedure of CSAM one can find exact analytical solutions for some particular Duffing equations without right hand side. Comparisons of exact solutions with solutions at the first approximation of CSAM, illustrate the accuracy of CSAM.

5. Procedure of CSAM can be used to general Duffing equations, the analytical approximate solutions obtained may be real valued, complex-valued or chaotric ones.

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