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AN APPROXIMATE SECULAR EQUATION OF RAYLEIGH WAVES IN AN ELASTIC HALF-SPACE COATED BY A THIN WEAKLY INHOMOGENEOUS ELASTIC LAYER

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Abstract. In this paper, the propagation of Rayleigh waves in a homogeneous isotropic elastic half-space coated with a thin weakly inhomogeneous isotropic elastic layer is investigated. The material parameters of the layer is assumed to depend arbitrarily continuously on the thickness variable. The contact between the layer and the half space is perfectly bonded. The main purpose of the paper is to establish an approximate secular equation of the wave. By applying the effective boundary condition method an approximate secular equation of second order in terms of the dimensionless thickness of the layer is derived. It is shown that the obtained approximate secular equation has good accuracy.

Keywords: Rayleigh waves, a homogeneous isotropic elastic half-space, a thin weakly inhomogeneous isotropic elastic layer, an approximate secular equation, the effective boundary condition method.

1. INTRODUCTION

The structures of a thin film attached to solids, modeled as half-spaces coated by a thin layer, are widely applied in modern technology. The determination of mechanical properties of thin films deposited on half-spaces before and during loading plays an important role in health monitoring of these structures [1,2]. Among various measurement methods, the surface/guided wave method is most widely used [2], and for this method the Rayleigh wave is a versatile and convenient tool [3,4].

For the Rayleigh-wave approach, the explicit dispersion relations of Rayleigh waves supported by thin-film/substrate interactions are employed as theoretical bases for extracting the mechanical properties of the thin films from experimental data. They are therefore the main purpose of any investigation of Rayleigh waves propagating in half-spaces covered with a thin layer. Taking the assumption of thin layer, explicit secular equations can be derived by replacing the entire effect of the thin layer on the half-space with *the so-called effective boundary conditions which relate the displacements with the stresses of the half-space at its surface*.

For obtaining the effective boundary conditions, Achenbach and Keshava [5], Tiersten [6] replaced the thin layer with a plate modeled by different theories: Mindlin's plate theory and the plate theory of low-frequency extension and flexure, while Bovik [7] expanded the stresses at the top surface of the layer into Taylor series in its thickness. The Taylor expansion technique was then developed by Rokhlin and Huang [8, 9], Niklasson [10], Benveniste [11], Steigmann and Ogden [12], Ting [13], Vinh and Linh [14, 15], Vinh and Anh [16, 17], Vinh et al. [18].

Achenbach and Keshava [5], Tiersten [6], Bovik [7] assumed that the layer and the substrate are both isotropic and derived approximate secular equations of second-order. In [12] Steigmann and Ogden considered a transversely isotropic layer with residual stress overlying an isotropic half-space and he obtained an approximate second order dispersion relation. In [19] Wang et al. considered an isotropic half-space covered by a thin electrode layer and the authors obtained an approximate secular equation of first order. In [14] the layer and the half-space were both assumed to be orthotropic and an approximate secular equation of third order was obtained. In [15] the layer and the half-space were both subjected to homogeneous pre-stains and an approximate secular equation of third order was established which is valid for any pre-strain and for a general strain energy function. In [17] the layer and the half-space are both isotropic and are perfectly bonded and an approximate dispersion relation of fourth order was established. In [16, 18] the layer and the half-space are in sliding contact and approximate secular equations of third order [18] and fourth order [16] were obtained.

In all investigations mentioned above, the layer and the half-space is assumed to be homogeneous. However, it is often the case that after being deposited to the half-space, the homogeneous layer becomes heterogeneous, as mentioned in [20], and the heterogeneity is usually weak. It is necessary to re-evaluate the mechanical properties of the deposited layer. The propagation of Rayleigh waves in half-spaces covered with a thin inhomogeneous layer is therefore becomes significant. In [20] the layer and the half-space are both isotropic and the material parameters of the layer continuously depend on the thickness coordinate. The inhomogeneity of the layer is assumed to be weak and its effect on the frequency dependence of the Rayleigh wave velocity was studied. At low-frequencies, an approximate formula of first order for the Rayleigh wave velocity was derived using the Peano expansion.

In this paper, we consider the propagation of Rayleigh waves in a homogeneous isotropic elastic half-space coated with a thin weakly inhomogeneous isotropic elastic layer. The material parameters of the inhomogeneous layer depend arbitrarily continuously on the thickness coordinate. The layer and the half space are in welded contact. The main purpose of the paper is to create an approximate secular equation of the wave. By applying the effective boundary condition method a second-order approximate secular equation in terms of the dimensionless thickness of the layer is derived. It is shown that the obtained approximate secular equation is a good approximation.

2. EFFECTIVE BOUNDARY CONDITION OF SECOND-ORDER

Consider a homogeneous, isotropic elastic half-space $x_3 \geq 0$ coated by a thin weakly inhomogeneous, isotropic elastic layer $-h \leq x_3 \leq 0$. The layer and the half-space are in perfectly bonded contact. We assumed that the material parameters of the inhomogeneous layer depend on only the thickness variable. In particular, the Lame constants $\bar{\lambda}(x_3)$, $\bar{\mu}(x_3)$ and the mass density $\bar{\rho}(x_3)$ of the layer are defined as

$$\begin{aligned} &\bar{\lambda} = \bar{\lambda}_0 f_1(z_1),\ \bar{\lambda}_0 := \bar{\lambda}(0),\ f_1(0) = 1 \\ &\bar{\mu} = \bar{\mu}_0 f_2(z_2),\ \bar{\mu}_0 := \bar{\mu}(0),\ f_2(0) = 1,\ z_k = \delta_k \frac{x_3}{h},\ k = 1,2,3 \\ &\bar{\rho} = \bar{\rho}_0 f_3(z_3),\ \bar{\rho}_0 := \bar{\rho}(0),\ f_3(0) = 1 \end{aligned} \tag{1}$$

where $f_k(.)$, $k = 1,2,3$ are arbitrary differentiable functions, δ_k, $k = 1,2,3$ are dimensionless parameters characterizing the inhomogeneity of the layer and being assumed to be small due to the assumption of weak inhomogeneity. Note that the same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer.

We consider a plane motion in the (x_1, x_3)-plane with displacement components (u_1, u_2, u_3) such that

$$u_i = u_i(x_1, x_3, t),\ \bar{u}_i = \bar{u}_i(x_1, x_3, t),\ i = 1,3,\ u_2 = \bar{u}_2 \equiv 0, \tag{2}$$

where t is the time. Since the layer is made of isotropic elastic materials, the strain-stress relations take the form

$$\begin{aligned} \bar{\sigma}_{11} &= (\bar{\lambda} + 2\bar{\mu})\bar{u}_{1,1} + \bar{\lambda}\bar{u}_{3,3}, \\ \bar{\sigma}_{33} &= \bar{\lambda}\bar{u}_{1,1} + (\bar{\lambda} + 2\bar{\mu})\bar{u}_{3,3}, \\ \bar{\sigma}_{13} &= \bar{\mu}(\bar{u}_{1,3} + \bar{u}_{3,1}), \end{aligned} \tag{3}$$

where $\bar{\sigma}_{ij}$ is the stress of the layer, commas indicate differentiation with respect to spatial variables x_k, $\bar{\lambda}$ and $\bar{\mu}$ are Lame constants. In the absent of body forces, the equations of motion for the layer is

$$\begin{aligned} \bar{\sigma}_{11,1} + \bar{\sigma}_{13,3} &= \bar{\rho}\ddot{\bar{u}}_1, \\ \bar{\sigma}_{13,1} + \bar{\sigma}_{33,3} &= \bar{\rho}\ddot{\bar{u}}_3, \end{aligned} \tag{4}$$

where a dot signifies differentiation with respect to t.

Now we consider the propagation of a Rayleigh wave, travelling (in the coated half-space) with velocity c (> 0) and wave number k (> 0) in the x_1-direction and decaying in the x_3-direction. The displacements and the stresses of the wave are sought in the form

$$\begin{aligned} &\bar{u}_1 = \bar{U}_1(x_3)e^{ik(x_1-ct)},\ \bar{u}_3 = \bar{U}_3(x_3)e^{ik(x_1-ct)}, \\ &\bar{\sigma}_{13} = ik\bar{T}_1(x_3)e^{ik(x_1-ct)},\ \bar{\sigma}_{33} = ik\bar{T}_3(x_3)e^{ik(x_1-ct)}, \end{aligned} \tag{5}$$

for the layer, and

$$
\begin{aligned}
u_1 &= U_1(x_3)e^{ik(x_1-ct)}, \quad u_3 = U_3(x_3)e^{ik(x_1-ct)}, \\
\sigma_{13} &= ikT_1(x_3)e^{ik(x_1-ct)}, \quad \sigma_{33} = ikT_3(x_3)e^{ik(x_1-ct)},
\end{aligned}
\tag{6}
$$

for the half-space. Form Eqs. (2)-(5), it is not difficult to verify that the unknown functions $\bar{U}_1$, $\bar{U}_3$, $\bar{T}_1$ and $\bar{T}_3$ satisfy the following matrix equation

$$\xi' = ikN\xi, \tag{7}$$

where

$$\xi = \begin{bmatrix} \bar{U} \\ \bar{T} \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \quad \bar{U} = \begin{bmatrix} \bar{U}_1 & \bar{U}_3 \end{bmatrix}^T \quad \bar{T} = \begin{bmatrix} \bar{T}_1 & \bar{T}_3 \end{bmatrix}^T \tag{8}$$

the symbol "T" indicate the transpose of a matrix, the prime signifies differentiation with respect to x_3 and

$$
\begin{aligned}
N_1 &= \begin{bmatrix} 0 & -1 \\ 2\bar{\gamma}-1 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} \dfrac{1}{\bar{\mu}} & 0 \\ 0 & \dfrac{1}{\bar{\lambda}+2\bar{\mu}} \end{bmatrix}, \\
N_3 &= \begin{bmatrix} 4(\bar{\gamma}-1)\bar{\mu}+\bar{\rho}c^2 & 0 \\ 0 & \bar{\rho}c^2 \end{bmatrix}, \quad N_4 = N_1^T,
\end{aligned}
\tag{9}
$$

where $\bar{\gamma} = \dfrac{\bar{\mu}}{\bar{\lambda}+2\bar{\mu}}$.

Let h be small (i.e the layer is thin), then expanding $\bar{T}(-h)$ into Taylor series at $x_3 = 0$ up to the second order of h we have

$$\bar{T}(-h) = \bar{T}(0) - \bar{T}'(0)h + \frac{1}{2!}\bar{T}''(0)h^2. \tag{10}$$

From (7) we have

$$\xi'' = ikN'\xi - k^2N^2\xi, \tag{11}$$

Suppose that surface $x_3 = -h$ of the layer is free of traction, i.e $\bar{T}(-h) = 0$. Using Eqs. (7), (8) and (11) at $x_3 = 0$ into (10) yields

$$
\begin{aligned}
&\Big[I - ikhN_4(0) + ik\frac{h^2}{2}N'_4(0) - \frac{k^2h^2}{2}N_6(0)\Big]\bar{T}(0) = \\
&= \Big[ikhN_3(0) - ik\frac{h^2}{2}N'_3(0) + \frac{k^2h^2}{2}N_5(0)\Big]\bar{U}(0),
\end{aligned}
\tag{12}
$$

where I is the identity matrix of order 2, matrices N_3 and N_4 are defined by (9) and

$$
\begin{aligned}
N'_3 &= \begin{bmatrix} 4(\bar{\gamma}-1)\bar{\mu}' + 4\bar{\mu}\bar{\gamma}' + \bar{\rho}'c^2 & 0 \\ 0 & \bar{\rho}'c^2 \end{bmatrix}, \quad N'_4 = \begin{bmatrix} 0 & 2\bar{\gamma}' \\ 0 & 0 \end{bmatrix}, \\
N_5 &= N_3N_1 + N_4N_3, \quad N_6 = N_3N_2 + N_4^2,
\end{aligned}
\tag{13}
$$

in which, according to Eqs. (1), $\bar{\gamma}'$, $\bar{\mu}'$ and $\bar{\rho}'$ and are calculated by

$$\bar{\mu}' = \frac{\bar{\mu}_0\delta_2 f_2'(z_2)}{h}, \ \bar{\rho}' = \frac{\bar{\rho}_0\delta_3 f_3'(z_3)}{h},$$
$$\bar{\gamma}' = \frac{\bar{\mu}_0\delta_2 f_2'(z_2)(\bar{\lambda}_0 + 2\bar{\mu}_0) - \bar{\mu}_0[\bar{\lambda}_0\delta_1 f_1'(z_1) + 2\bar{\mu}_0\delta_2 f_2'(z_2)]}{(\bar{\lambda}_0 + 2\bar{\mu}_0)^2}\frac{1}{h}, \tag{14}$$

Since the layer and the half-space are bonded perfectly to each other at the plane $x_3 = 0$, it follows $\bar{U}(0) = U(0)$ and $\bar{T}(0) = T(0)$. Thus, from Eq. (12) we arrive at

$$\begin{aligned}\Big[I - ikhN_4(0) + ik\frac{h^2}{2}N'_4(0) - \frac{k^2h^2}{2}N_6(0)\Big]T(0) = \\ = \Big[ikhN_3(0) - ik\frac{h^2}{2}N'_3(0) + \frac{k^2h^2}{2}N_5(0)\Big]U(0).\end{aligned} \tag{15}$$

The relation (15) between the traction vector and displacement vector of the half-space at its surface $x_3 = 0$ is called the effective boundary condition of second order in the matrix form. Substituting (9) and (13) into (15) yields the effective boundary conditions in the components form, namely,

$$\begin{aligned}&T_1(0) + i\varepsilon\Big\{(1 - 2\bar{\gamma}_0 + h\bar{\gamma}_0')T_3(0) + \big[4(1-\bar{\gamma}_0) - r_v^2 x\big]\bar{\mu}_0 U_1(0) + h\Big[2(\bar{\gamma}_0 - 1)\bar{\mu}_0' + 2\bar{\mu}_0\bar{\gamma}_0' \\ &+ \frac{\bar{\rho}_0' c^2}{2}\Big]U_1(0)\Big\} + i\frac{\varepsilon^2}{2}\Big\{(2\bar{\gamma}_0 - 3 + r_v^2 x)iT_1(0) + \big[4(1-\bar{\gamma}_0)\bar{\mu}_0 + 2(\bar{\gamma}_0 - 1)\bar{\rho}_0 c^2\big]iU_3(0)\Big\} = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&T_3(0) + i\varepsilon\Big[T_1(0) + \Big(\frac{h}{2}\bar{\rho}_0' c^2 - \bar{\rho}_0 c^2\Big)U_3(0)\Big] + i\frac{\varepsilon^2}{2}\Big\{(1 - 2\bar{\gamma}_0 + r_v^2\bar{\gamma}x)iT_3(0) \\ &+ [4(1-\bar{\gamma}_0)\bar{\mu}_0 + 2(\bar{\gamma}_0 - 1)\bar{\rho}_0 c^2]\bar{\mu}_0 iU_1(0)\Big\} = 0,\end{aligned} \tag{17}$$

where $\varepsilon = kh$ is the dimensionless thickness of the layer, and

$$r_v = \frac{c_2}{\bar{c}_{20}}, \ c_2 = \sqrt{\frac{\mu}{\rho}}, \ \bar{c}_{20} = \sqrt{\frac{\bar{\mu}_0}{\bar{\rho}_0}}, \ x = \frac{c^2}{c_2^2}, \ 0 < x < 1 \tag{18}$$

Here we use the notations: $\bar{\lambda}_0' := \bar{\lambda}'(0)$, $\bar{\mu}_0' := \bar{\mu}'(0)$, $\bar{\rho}_0' := \bar{\rho}'(0)$.

3. AN APPROXIMATE SECULAR EQUATION OF SECOND ORDER

Now we can ignore the layer and consider the propagation of Rayleigh waves in the isotropic elastic half-space $x_3 \geq 0$ whose surface $x_3 = 0$ is subjected to the boundary conditions (16), (17). According to Achenbach [21], the displacement components of a Rayleigh wave travelling with velocity c and wave number k in the x_1-direction and decaying in the x_3-direction are determined by $(6)_{1,2}$ in which $U_1(x_3)$ and $U_3(x_3)$ are given by

$$\begin{aligned}U_1(x_3) &= B_1 e^{-b_1 k x_3} + B_2 e^{-b_2 k x_3}, \\ U_3(x_3) &= \alpha_1 B_1 e^{-b_1 k x_3} + \alpha_2 B_2 e^{-b_2 k x_3},\end{aligned} \tag{19}$$

where B_1 and B_2 are constant to be determined and

$$b_1 = \sqrt{1-\gamma x},\ b_2 = \sqrt{1-x},\ \alpha_1 = -\frac{b_1}{i},\ \alpha_2 = \frac{i}{b_2},\ \gamma = \frac{\mu}{\lambda+2\mu}.$$

Substituting $(6)_{1,2}$ and (19) into the stress-strain relations (3) without the bar yields that the stresses σ_{13} and σ_{33} are given by $(6)_{3,4}$ in which

$$\begin{aligned} T_1(x_3) &= i\mu\big[(b_1+\beta_1)B_1e^{-b_1kx_3} + (b_2+\beta_2)B_2e^{-b_2kx_3}\big],\\ T_3(x_3) &= \mu\Big[\Big(\frac{1}{\gamma}-2-\frac{1}{\gamma}b_1\beta_1\Big)B_1e^{-kb_1x_3} + \Big(\frac{1}{\gamma}-2-\frac{1}{\gamma}b_2\beta_2\Big)B_2e^{-kb_2x_3}\Big], \end{aligned} \tag{20}$$

where $\beta_k = -i\alpha_k$, $k = 1, 2$. Introducing (19) and (20) into the effective boundary conditions (16) and (17) leads to the following equations for B_1, B_2

$$\begin{cases} f(b_1)B_1 + f(b_2)B_2 = 0\\ F(b_1)B_1 + F(b_2)B_2 = 0 \end{cases} \tag{21}$$

where

$$\begin{aligned} f(b_n) &= (b_n+\beta_n) + \varepsilon\Big\{(1-2\bar\gamma_0 + h\bar\gamma_0')\Big(\frac{1}{\gamma}-2-\frac{1}{\gamma}b_n\beta_n\Big) + r_\mu(4-4\bar\gamma_0 - r_v^2x)\\ &\quad + h[2(\bar\gamma_0-1)\frac{\bar\mu_0'}{\mu} + 2r_\mu\bar\gamma_0' + \frac{1}{2}\frac{\bar\rho_0'}{\rho}x]\Big\} + \frac{\varepsilon^2}{2}\Big[(3-2\bar\gamma_0 - r_v^2x)(b_n+\beta_n)\\ &\quad - 2r_\mu(1-\bar\gamma_0)(2-r_v^2x)\beta_n\Big],\\ F(b_n) &= \Big(\frac{1}{\gamma}-2-\frac{1}{\gamma}b_n\beta_n\Big) + \varepsilon\Big[-(b_n+\beta_n) + r_\mu r_v^2x\beta_n - \frac{1}{2}h\frac{\bar\rho_0'}{\rho}x\beta_n\Big]\\ &\quad + \frac{\varepsilon^2}{2}\Big[(2\bar\gamma_0 - 1 - r_v^2\bar\gamma_0x)\Big(\frac{1}{\gamma}-2-\frac{1}{\gamma}b_n\beta_n\Big) - 2r_\mu(1-\bar\gamma_0)(2-r_v^2x)\Big], \end{aligned} \tag{22}$$

in which $r_\mu = \dfrac{\bar\mu_0}{\mu}$. For a non-trivial solution, the determinant of the system (21) must vanish. This provides

$$f(b_1)F(b_2) - f(b_2)F(b_1) = 0. \tag{23}$$

Using (22) into (23), we arrive at the approximate secular equation of second order in terms of ε of Rayleigh waves, namely,

$$A_0 + (A_1 + \bar A_1)\varepsilon + (A_2 + \bar A_2 + \hat A_2)\frac{\varepsilon^2}{2} = 0, \tag{24}$$

where

$$
\begin{aligned}
A_0 &= (2-x)^2 - 4b_1b_2,\\
A_1 &= xr_\mu\Big\{[r_v^2x - 4(1-\bar\gamma_0)]b_2 + r_v^2xb_1\Big\},\\
\bar A_1 &= h\Big\{-\frac{1}{2}\frac{\bar\rho_0'}{\rho}x^2b_1 + xb_2\Big[2(1-\bar\gamma_0)\frac{\bar\mu_0'}{\mu} - 2r_\mu\bar\gamma_0' - \frac{1}{2}\frac{\bar\rho_0'}{\rho}x\Big]\Big\},\\
A_2 &= -A_0\big[xr_v^2(1+\bar\gamma_0) - 4(1-\bar\gamma_0)\big] + 2r_\mu^2r_v^2x\big[4(1-\bar\gamma_0) - r_v^2x\big](1-b_1b_2)\\
&\quad + 2r_\mu(2b_1b_2 - 2 + x)\big[4(1-\bar\gamma_0) - 2r_v^2x\bar\gamma_0\big],\\
\bar A_2 &= h\Big\{2\bar\gamma_0'A_0 + 2\frac{\bar\rho_0'}{\rho}\bar\gamma_0x^2 + 2r_\mu r_v^2\bar\gamma_0'x^2 + 4(\bar\gamma_0 - 1)\frac{\bar\mu_0'}{\mu}x + 4r_\mu\bar\gamma_0'x\\
&\quad + \Big[2\frac{\bar\rho_0'}{\rho}x[r_\mu r_v^2x - 2\bar\gamma_0 - 2r_\mu(1-\bar\gamma_0)] + 4r_\mu\bar\gamma_0'[r_v^2x(r_\mu - 1) - 2]\\
&\quad + 4\frac{\bar\mu_0'}{\mu}(1-\bar\gamma_0)(2 - r_\mu r_v^2x)\Big](1-b_1b_2)\Big\},\\
\hat A_2 &= h^2\Big\{\frac{\bar\rho_0'}{2\rho}x\Big[4\bar\gamma_0'(1-r_\mu) + 4(1-\bar\gamma_0)\frac{\bar\mu_0'}{\mu} - \frac{\bar\rho_0'}{\rho}x\Big](1-b_1b_2) - \frac{\bar\rho_0'}{\rho}\bar\gamma_0'x^2\Big\},
\end{aligned}
\tag{25}
$$

in which, from (14), the quantities $h\bar\gamma_0'$, $h\bar\mu_0'$ and $h\bar\rho_0'$ in the expressions of $\bar A_1$, $\bar A_2$ and $\hat A_2$ are given by

$$
\begin{aligned}
h\bar\mu_0' &= \bar\mu_0\delta_2 f_2'(0),\ h\bar\rho_0' = \bar\rho_0\delta_3 f_3'(0),\\
h\bar\gamma_0' &= \frac{\bar\mu_0\delta_2 f_2'(0)(\bar\lambda_0 + 2\bar\mu_0) - \bar\mu_0[\bar\lambda_1\delta_1 f_1'(0) + 2\bar\mu_0\delta_2 f_2'(0)]}{(\bar\lambda_0 + 2\bar\mu_0)^2}.
\end{aligned}
\tag{26}
$$

Eq. (24) is the desired second-order approximate secular equation. This equation is totally explicit. Left-hand side of Eq. (24) is an explicit function of x (the squared dimensionless Rayleigh wave velocity), γ, $\bar\gamma$, r_μ, r_v (the dimensionless material parameters of the layer and the half-space), δ_1, δ_2, δ_3 (the inhomogeneity dimensionless parameters) and ε (the dimensionless thickness of the layer), provided the functions $f_1(z_1)$, $f_2(z_2)$ and $f_3(z_3)$ are given.

Fig. 1 presents the exact curve (solid line) and the second-order approximate curve (24) (dashed line) of the dimensionless Rayleigh wave velocity $\sqrt{x} = c/c_2$. Here we take

$\mu = 2.85 \times 10^{-10}\ \mathrm{N/m^2}, \lambda = 12.112 \times 10^{-10}\ \mathrm{N/m^2}, \rho = 3.1667 \times 10^{-17}\ \mathrm{kg/m^3}$,

$\bar\mu = 3.1 \times 10^{-10}e^{\delta z}\ \mathrm{N/m^2}, \bar\lambda = 10.5 \times 10^{-10}e^{\delta z}\ \mathrm{N/m^2}, \bar\rho = 3.1 \times 10^{-16}e^{\delta z}\ \mathrm{kg/m^3}$,

$\delta = 0.1, z = \dfrac{x_3}{h}$, i.e.: $\delta_k = \delta = 0.1$, $z_k = \delta z$, $f_k(z_k) = e^{z_k}, k = 1,2,3$.

It implies: $\gamma = 0.16$, $\bar\gamma = 0.1856$, $r_\mu = 1.0877$, $r_v = 3$, $\delta_1 = \delta_2 = \delta_3 = 0.1$. In Fig. 2, the material parameters are taken as in Fig. 1, except $\delta = -0.1$. Note that to draw the exact wave velocity curves we replace the inhomogeneous layer by N (being sufficiently large) homogeneous layers perfectly bonded to each other, and then apply the transfer matrix method [22, 23]. It is seen from Figs. 1, 2 that the second-order approximate velocity curves are close to the corresponding exact velocity curves in the interval $\varepsilon \in$

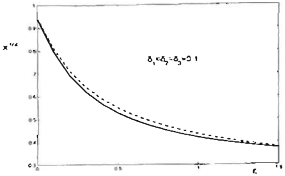

Fig. 1. The exact curve (solid line) and the second-order approximate curve (24) (dashed line) of the dimensionless Rayleigh wave velocity $\sqrt{x} = c/c_2$ ($\delta = 0.1$)

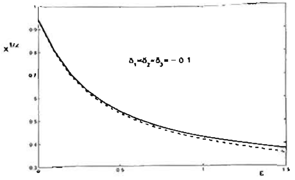

Fig. 2. The exact curve (solid line) and the second-order approximate curve (24) (dashed line) of the dimensionless Rayleigh wave velocity $\sqrt{x} = c/c_2$ ($\delta = -0.1$)

[0 1.5]. This says that the obtained approximate secular equation has good accuracy. For $\delta = 0.1$ the approximate velocity curve lies above the exact velocity curve, while for $\delta = -0.1$ the approximate velocity curve lies below the exact one.

When the layer is homogeneous we have: $\delta_1 = \delta_2 = \delta_3 = 0$, it implies from (14): $\bar{\gamma}'_0 = \bar{\mu}'_0 = \bar{\rho}'_0 = 0$. From these fact and (25) it follows $\bar{A}_1 = \bar{A}_2 = \hat{A}_2 = 0$. The equation (24) is thus simplified to

$$A_0 + A_1\varepsilon + \frac{A_2}{2}\varepsilon^2 = 0, \tag{27}$$

that coincides with the approximate secular equation of second order of Rayleigh waves propagating in an isotropic elastic half-space covered with a thin homogeneous isotropic elastic layer, Eq. (40) in Ref. [14].

When the layer is absent, i.e.: $\varepsilon = 0$, Eq. (24) is simplified to $A_0 = 0$. By $(25)_1$ it is

$$(2 - x)^2 - 4\sqrt{1 - x}\,\sqrt{1 - \gamma x} = 0. \tag{28}$$

This is the secular equation of Rayleigh waves propagating along the traction-free surface of a compressible isotropic elastic half-space that was obtained by Rayleigh [24] in 1885. Although this equation was discovered nearly 130 years ago and the explicit analytical expressions of its (unique) solution corresponding to Rayleigh waves are significant in many practical applications, they were derived only recently [25, 26]. Before the appearance of the explicit analytical formulas for Rayleigh waves velocity, the approximate Rayleigh wave velocity formulas have been established, see Refs. [27–30]. Their accuracy were improved recently, see Refs. [31–33], by using the obtained explicit analytical formulas for Rayleigh waves velocity along with the method of least squares.

4. CONCLUSIONS

In this paper, the propagation of Rayleigh waves in a homogeneous isotropic elastic half-space coated with a thin weakly inhomogeneous isotropic elastic layer is investigated. The material parameters of the layer is assumed to depend arbitrarily continuously on the thickness variable. The contact between the layer and the half space is perfectly bonded. An approximate secular equation of second order in the dimensionless thickness of the layer is derived by using the effective boundary condition method. It is shown that the obtained approximate secular equation has good accuracy. Since the obtained secular equation is totally explicit, it is a good tool for extracting the mechanical properties of the thin films from experimental data.

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