THE EXISTENCE OF BOUNDED SOLUTIONS TO THE COMPLEX MONGE-AMPÈRE TYPE EQUATIONS ON HERMITIAN MANIFOLDS WITH BOUNDARY

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Abstract: *In the paper, we will study and discuss about the existence of bounded solution to the complex Monge-Ampère type equation $F(u,z)d\mu = (\omega + dd^c u)^n$ on Hermitian manifold $(\overline{X},\omega)$ with nonempty boundary, where u is an unknown function, μ is a nonnegative Radon measure on X and $F(t,z)$ is a $dt \times d\mu$ - measurable function.*

Keywords: *bounded solutions, bounded subsolutions, complex Monge-Ampère type equation, Hermitian manifolds, ω-plurisubharmonic functions.*

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1. INTRODUCTION

Let $(\overline{X},\omega)$ be a Hermitian manifold with $dim_{\mathbb{C}} X = n \geq 1$, nonempty boundary ∂X and a fixed background Hermitian metric ω on $\overline{X}$. In this paper, we always assume that the Hermitian metric ω satisfies the following conditions:

$$-B\omega^2 \leq 2ndd^c\omega \leq B\omega^2; -B\omega^3 \leq 4n^2 d\omega \wedge d^c\omega \leq B\omega^3 \quad (1)$$

in $\overline{X}$, where, $B > 0$ only depends on ω, n; we use differential operators $d = \partial + \overline{\partial}$, $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$; then $dd^c = \frac{i}{\pi}\partial\overline{\partial}$ $dd^c = \frac{i}{\pi}\partial\,\bar{\partial}$.

By $PSH(\omega, X)$ we denote set of ω -plurisubharmonic functions on X. Let μ be a nonnegative Radon measure on X. Assume that $F: \mathbb{R} \times X \to [0, +\infty)$ is a $dt \times d\mu$ -measurable function. We consider the following Monge-Ampère type equation

$$\begin{cases} u \in PSH(\omega, X) \cap L^\infty(X); \\ (\omega + dd^c u)^n = F(u,\cdot)d\mu; \end{cases} \quad (2)$$

where u is an unknown function and $(\omega + dd^c u)^n$ stands for the complex Monge-Ampère operator of u.

When X is a bounded domain in $\mathbb{C}^n$ and ω is the zero form, the equation (2) is the Monge-Ampère type equations for plurisubharmonic functions in X. In this case, solutions of (2) are not only considered in the class of bounded functions but also extended to unbounded functions. There are many results on solving this equation due to [1], [2], [3], [12], etc.

On a Hermitian manifold $(\overline{X},\omega)$ with nonempty boundary, if $F(t,z) \in C^\infty(\mathbb{R} \times X)$ and $\mu = \omega^n$ such that $F(v,z) \leq (\omega + dd^c v)^n$ with some smooth ω -plurisubharmonic function v in X, the authors in [6] proved that the equation (2) exists a smooth solution. When $F(t,z) = f(z) \in L^p(X), p > 1$ and μ is a smooth measure, the authors in [10] showed that there exists a continuous solution of the equation (2). Recently, the authors in [7] and [8] obtained some results on the existence of bounded solution of the equation (2). Namely, when $F(t,z) = 1$and $\mu \leq (\omega + dd^c v)^n$, with some $v \in PSH(\omega, X) \cap L^\infty(X)$ such that the restriction of v in ∂X is a continuous function, the authors in [7] proved that the euquation (2) has a bounded solution u such that $u = v$ in ∂X. In the case that $F(t,z)$ is bounded, the authors in [8] also showed the existence of bounded solution of (2) when $F(t,z)$ satisfies that there exists a function $v \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(v,\cdot)d\mu \leq (\omega + dd^c v)^n$. Also in [8], the authors gave the proof of the existence of bounded solution to the equation (2) when $F(t,z) = e^{\lambda t}$ and $\mu \leq (\omega + dd^c v)^n$ with some $v \in PSH(\omega, X) \cap L^\infty(X)$. Continuing the above direction of research, in this paper, we wish to investigate the existence of bounded solutions to (2) in case of the arbitrary function $F(t,z)$.

Now we give the main result of our paper which is inspired from the results in [3] and [8].

Theorem Let $(\overline{X} = X \cup \partial X, \omega)$ be an n -dimensional Hermitian manifold such that the boundary ∂X is nonempty and there exists a function $f \in PSH(\omega, X) \cap C(X)$ satisfying that $(\omega + dd^c f)^n = 0$. Let μ be a finite non-negative Radon measure in X satisfying that at each point $z \in X$, there exist a neighborhood U_z of z and a function $v_z \in PSH(U_z) \cap L^\infty(U_z)$ such that $\mu \leq (dd^c v_z)^n$. Assume that $F: \mathbb{R} \times X \to [0, +\infty)$ is a $dt \times d\mu$ -measurable function such that:
(1) For all $z \in X$, the function $t \mapsto F(t,z)$ is continuous and non-decreasing.
(2) At each $z \in X$, there exist a neighborhood W_z of z and a function $w_z \in PSH(W_z) \cap L^\infty(W_z)$ such that $F(t,\cdot)d\mu \leq (dd^c w_z)^n, \forall t \in \mathbb{R}$, in W_z.
(3) There exists $u \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(u,\cdot)d\mu \leq (\omega + dd^c u)^n$ and $u \leq f$.

Then there exists a function $w \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(w,\cdot)d\mu = (\omega + dd^c w)^n$ and $w \leq f$.

The paper is organized as follows. In Section 2, we recall some notions of ω -plurisubharmonic functions which are necessary for the next results of the paper; after we give the proof of Theorem 1.1. Section 3 gives conclusion of paper.

2. CONTENT

2.1. Preliminaries

In this section, we recall some elements of theory of ω -plurisubharmonic functions that will be used throughout the paper. These results can be found in [4], [5], [9], [10]. In this paper, by $PSH(\Omega)$ we denote set of plurisubharmonic functions in open set $\Omega \subset \mathbb{C}^n$. By $C(X)$ we denote class of continuous functions in X.

Now, we recall the definition ofω -plurisubharmonic functions in open subsets of $\mathbb{C}^n$. Let Ω be an open subset in $\mathbb{C}^n$ and ω be a Hermitian metric on $\mathbb{C}^n$. We have the following definition.

Definition 2.1. Let a function $u: \Omega \to [-\infty, +\infty)$ be an upper semi-continuous function. Then u is called ω -plurisubharmonic (ω -psh for short) if $u \in L^1_{loc}(\omega^n, \Omega)$ and $\omega + dd^c u \geq 0$ as a current. By $PSH(\omega, \Omega)$ we denote set of ω -psh functions in Ω.

Note that for each $u \in PSH(\omega, \Omega)$ and each point $z \in \Omega$, there exist a neighborhood $U \subset \Omega$ of z and a smooth plurisubharmonic function ρ in U such that $\omega \leq dd^c \rho$ in U and $u + \rho \in PSH(U)$.

Next, let (X, ω) be an n -dimensional Hermitian manifold. We have the following definition.

Definition 2.2. An upper semi-continuous function $u: X \to [-\infty, +\infty)$ is called ω -plurisubharmonic if $u \in L^1_{loc}(\omega^n, X)$ and $\omega + dd^c u \geq 0$ as a current. By $PSH(\omega, X)$ we also denote set of ω -psh functions in X.

Note that $u \in PSH(\omega, X)$ if and only if $u \in PSH(\omega, \Omega)$ for arbitrary coordinate chart $\Omega \subset X$.

Finally, we give the introduction of the complex Monge - Ampère operator of ω - plurisubharmonic functions. Fix an open subset $\Omega \subset \mathbb{C}^n$ and a Hermitian metric ω on $\mathbb{C}^n$. Let $u_1, \ldots, u_k \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$, for $k \in [1, n]$. By the arguments in [4] and [10], the wedge product $(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_k)$ is defined as a nonnegative (k, k) -current in Ω.

When $k = n$ and $u_1 = \cdots = u_n = u$, the wedge product

$$(\omega + dd^c u)^n := \underbrace{(\omega + dd^c u) \wedge \cdots \wedge (\omega + dd^c u)}_{n \quad times}$$

is called complex Monge-Ampère operator of u. This operator is also defined as a nonnegative Radon measure in Ω Ω. Furthermore, if $u_j \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$ is either uniformly convergent or monotonely convergent almost everywhere to $u \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$, then $(\omega + dd^c u_j)^n \to (\omega + dd^c u)^n$ in the sense of currents.

Now, let $(\overline{X}, \omega)$ be an n -dimensional Hermitian manifold. Take $u \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$. Then $u \in PSH(\omega, \Omega)$ for any coordinate chart $\Omega \subset\subset X$. By using partition of unity, we define the complex Monge-Ampère operators $(\omega + dd^c u)^n$ of u. Moreover, $(\omega + dd^c u)^n$ is a nonnegative Radon measure in X. It is also clear that $(\omega + dd^c u_j)^n \to (\omega + dd^c u)^n$ in the sense of currents for every sequence $\{u_j\} \subset PSH(\omega, X) \cap L^\infty(X)$ satisfying that u_j converges either uniformly or monotonely a.e. to $u \in PSH(\omega, X) \cap L^\infty(X)$.

2.2. The existence of bounded solutions to the complex Monge-Ampère type equations

In this section, we will establish a result on the sovablity of the equation (2). Namely, the content of this result is expressed by Theorem 1.1 in Section 1 and we will give its proof here.

Firstly, we give a local result which is inspired form the results in [3] and [8] and used for the proof of Theorem 1.1. Fix a strictly pseudoconvex domain Ω in $\mathbb{C}^n$. Fix a Hermitian metric ω in neighborhood U of $\overline{\Omega}$ such that in U, the condition (1) is still valid for ω. Let φ be a continuous function on $\partial\Omega$. Theorem 4.2 and Corollary 3.4 in [10] follows that the following equation

$$\begin{cases} h \in PSH(\omega, \Omega) \cap L^\infty(\Omega); \\ (\omega + dd^c h)^n = 0, \qquad in \quad \Omega; \\ h = \varphi, \qquad on \quad \partial\Omega; \end{cases}$$

always exists a unique solution h. For convenience, by h_φ we denote solution of the above equation. We have the following lemma.

Lemma 2.3. Let a finite nonnegative Radon measure μ be such that $\mu \le (dd^c v)^n$ in Ω, for some $v \in PSH(\Omega) \cap L^\infty(\Omega)$. Let φ be a continuous function on $\partial\Omega$. Let $F: \mathbb{R} \times \Omega \to [0, +\infty)$ be a $dt \times d\mu$ -measurable function such that:

(1) For all $z \in \Omega$, the function $t \mapsto F(t, z)$ is continuous, nondecreasing.
(2) For all $t \in \mathbb{R}$, the function $z \mapsto F(t, z)$ is in $L^1(d\mu, \Omega)$.
(3) There exists $w \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$ such that $F(h_\varphi, \cdot) d\mu \le (\omega + dd^c w)^n$ and $w = \varphi$ on $\partial\Omega$.

Then, there exists a unique function $u \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$ such that $F(u,\cdot)d\mu = (\omega + dd^c u)^n$ and $u = \varphi$ on $\partial\Omega$.

Proof. Firstly, we will prove lemma when $v \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $v = 0$ on $\partial\Omega$ and $\int_\Omega (dd^c v)^n < +\infty$. From the hypotheses of $F(t,z)$ and the definition of h_φ and by using Corollary 3.4 in [10], we get that $w \le h_\varphi$. Set

$$\mathcal{A} = \{\psi \in PSH(\omega, \Omega): \quad w \le \psi \le h_\varphi\}.$$

It is easy to show that $\mathcal{A}$ is convex and compact. Fix each function $\psi \in \mathcal{A}$. Then we get that

$$F(\varphi,\cdot)d\mu \le F(h_\varphi,\cdot)d\mu \le (\omega + dd^c w)^n.$$

Thus, Theorem 1.2 in [7] and Corollary 3.4 in [10] follow that there exists a unique function $\tilde{\psi} \in \mathcal{A}$ such that $F(\psi,\cdot)d\mu = (\omega + dd^c\tilde{\psi})^n$. So, we can define an operator $T: \mathcal{A} \to \mathcal{A}$ such that $T(\psi) = \tilde{\psi}$, for every $\psi \in \mathcal{A}$.

Next, we will claim that T is continuous. Indeed, let an arbitrary sequence $\{\psi_j\} \subset \mathcal{A}$ and $\psi \in \mathcal{A}$ be such that ψ_j converges to ψ in $L^1(\Omega)$. We will prove that $T(\psi_j)$ is convergent to $T(\psi)$ in $L^1(\Omega)$. Set $\widetilde{\psi_j} = T(\psi_j)$ and $\tilde{\psi} = T(\psi)$. Take a cluster point $\hat{\psi} \in PSH(\omega, \Omega)$ of $\{\widetilde{\psi_j}\}$. It is sufficient to show that $\hat{\psi} = \tilde{\psi}$. Since $\{\widetilde{\psi_j}\} \subset \mathcal{A}$ and $\mathcal{A}$ is compact, then we get that $\hat{\psi} \in \mathcal{A}$ and there exists a subsequence of $\{\widetilde{\psi_j}\}$, still denoted by $\{\widetilde{\psi_j}\}$, such that $\widetilde{\psi_j}$ converges to $\hat{\psi}$ in $L^1(\Omega)$. Since ψ_j converges in $L^1(\Omega)$ to ψ, then Lemma 2.1 and Corollary 2.2 in [8] follows that we can extract a subsequence (still denoted by $\{\psi_j\}$) such that ψ_j converges to ψ in $L^1(d\mu, \Omega)$ and $L^1(\Omega)$. Passing to a subsequence, we get that ψ_j is convergent to ψ a.e. w.r.t. μ. Hence we obtain that $F(\psi_j,\cdot)$ converges a.e. w.r.t. μ to $F(\psi,\cdot)$. Then Lebesgue-dominated convergence theorem implies that $F(\psi_j,\cdot)d\mu$ converges weakly to $F(\psi,\cdot)d\mu$.

By the hypotheses of $F(t,z)$ and $F(h_\varphi,\cdot) \in L^1(d\mu, \Omega)$, it follows that there exists a unique function $v_1 \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $v_1 = 0$ on $\partial\Omega$, $\int_\Omega (dd^c v_1)^n < +\infty$ and $F(h_\varphi,\cdot)d\mu = (dd^c v_1)^n$. Since $F(\psi_j,\cdot)d\mu \le (dd^c v_1)^n$ and $F(\psi,\cdot)d\mu \le (dd^c v_1)^n$, Radon-Nikodym theorem follows that $F(\psi_j,\cdot)d\mu = f_j(dd^c v_1)^n$ and $F(\psi,\cdot)d\mu = f(dd^c v_1)^n$, where $0 \le f_j, f \le 1$.

On the other hand, since $(\omega + dd^c\widetilde{\psi_j})^n = f_j(dd^c v_1)^n$ and $f_j(dd^c v_1)^n$ is weakly convergent to $f(dd^c v_1)^n$, then Theorem 2.7 in [8] implies that $(\omega + dd^c\hat{\psi})^n = f(dd^c v_1)^n$. By applying Corollary 3.4 in [10] and $(\omega + dd^c\tilde{\psi})^n = f(dd^c v_1)^n$, we get that $\tilde{\psi} = \hat{\psi}$. It follows that $T(\psi_j)$ converges in $L^1(\Omega)$ to $T(\psi)$ and we get the desired conclusion.

Since T continues, then Schauder's fixed point theorem and Proposition 2.2 in [8] imply that there exists a unique function $u \in \mathcal{A}$ such that $T(u) = u$. Hence, we get the desired conclusion.

We now complete the proof when $v \in PSH(\Omega) \cap L^\infty(\Omega)$. Take a smooth plurisubharmonic function ρ such that $\Omega = \{\rho < 0\}, \partial\Omega = \{\rho = 0\}$ and $\nabla\rho \ne 0$ on $\partial\Omega$. Let $\{\Omega_k\}$ be a increasing sequence of strictly pseudoconvex domains such that $\overline{\Omega_k} \subset\subset \Omega_{k+1} \subset\subset \Omega, \forall k \ge 1$ and $\bigcup_{k=1}^{+\infty} \Omega_k = \Omega$. Set $w_k = max(M_k\rho, v)$, where $M_k > 0$ satisfies that $M_k\rho \le v$ on $\overline{\Omega_k}$. Then $\mu_k := 1_{\Omega_k}\mu \le (dd^c w_k)^n$. Since $\mu_k(\Omega) < +\infty$, Kołodziej's subsolution theorem in [11] follows that $\mu_k = (dd^c v_k)^n$ with $v_k \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $w_k \leq v_k \leq 0$ and $\int_\Omega (dd^c v_k)^n < +\infty$.

Now, we apply the first part for $F(t,z)$ and μ_k. Then there exists a unique function $u_k \in PSH(\omega, \Omega)$ such that $F(u_k,\cdot)d\mu = (\omega + dd^c u_k)^n$ and $w \leq u_k \leq h_\varphi$.

Since $\mu_k \leq \mu_{k+1}$, then Proposition 2.2 in [8] follows that $u_{k+1} \leq u_k$. Set $u = \lim_{k\to\infty} u_k$. Then $u \in PSH(\omega, \Omega)$ and $w \leq u \leq h_\varphi$. Since u_k decreases pointwise to u in Ω, then we obtain that $(\omega + dd^c u_k)^n$ converges weakly to $(\omega + dd^c u)^n$. By applying Lebesgue-dominated convergence theorem, we get that in each Ω_j for $j \geq 1$, $F(u_k,\cdot)d\mu$ is weakly convergent to $F(u,\cdot)d\mu$ as k tends to ∞. It follows that $F(u_k,\cdot)d\mu$ is weakly convergent to $F(u,\cdot)d\mu$ in Ω. Hence, we get that $F(u,\cdot)d\mu = (\omega + dd^c u)^n$. The uniqueness of u is implied by Proposition 2.2 in [8]. The proof is complete. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Set

$$w = (sup\{\varphi \in PSH(\omega, X) \cap L^\infty(X): \quad \varphi \leq f \quad on \quad X \quad and \quad (\omega + dd^c\varphi)^n \geq F(\varphi,\cdot)d\mu \quad on \quad X \quad \})^*.$$

By the hypotheses and the definition of w, we infer that $u \leq w \leq f$ and $w \in PSH(\omega, X)$. On the other hand, we have the following estimate

$$F(max(\varphi,\psi),\cdot)d\mu \leq (\omega + dd^c\, max(\varphi,\psi))^n, \quad \textbf{(3)}$$

in X, for every $\varphi, \psi \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(\varphi,\cdot)d\mu \leq (\omega + dd^c\varphi)^n$ and $F(\psi,\cdot)d\mu \leq (\omega + dd^c\psi)^n$ in X.

Now, we will claim that

$$F(w,\cdot)d\mu \leq (\omega + dd^c w)^n. \quad \textbf{(4)}$$

Indeed, Choquet's Lemma and the inequality (3) follows that we can choose an increasing sequence $\{\varphi_j\} \subset PSH(\omega, X) \cap L^\infty(X)$ such that $u \leq \varphi_j \leq f$ on X, $\varphi_j \nearrow w$ a.e on X and

$$F(\varphi_j,\cdot)d\mu \leq \left(\omega + dd^c\varphi_j\right)^n. \quad \textbf{(5)}$$

Since the increasing sequence $\{\varphi_j\}$ of bounded ω -plurisubharmonic functions converges a.e. to w in X, then we obtian that $(\omega + dd^c\varphi_j)^n$ is weakly convergent to $(\omega + dd^c w)^n$ as $j \to \infty$.

On the other hand, for an arbitrary coordinate ball $\boldsymbol{B} \subset\subset X$ such that $\mu \leq (dd^c v)^n$ and $F(t,\cdot)d\mu \leq (dd^c v)^n, \forall t \in \mathbb{R}$, where v is a bounded plurisubharmonic function in some neighborhood of $\overline{\boldsymbol{B}}$, then Lemma 2.1 and Corollary 2.2 in [8] follows that there exists a subsequence of $\{\varphi_j\}$ (still denoted by $\{\varphi_j\}$) such that φ_j converges to w in $L^1(d\mu, \boldsymbol{B})$. Then $F(\varphi_j,\cdot)d\mu$ weakly converges in $\boldsymbol{B}$ to $F(w,\cdot)d\mu$ as $j \to \infty$. It implies that $F(\varphi_j,\cdot)d\mu$ is weakly convergent in X to $F(\varphi,\cdot)d\mu$. Thus, from the inequality (5) we get that the inequality (4) is true.

Next, we fix an arbitrary coordinate ball $\boldsymbol{B} \subset\subset X$ such that $\mu \leq (dd^c v)^n$ and $F(t,\cdot)d\mu \leq (dd^c v)^n, \forall t \in \mathbb{R}$, with $v \in PSH(\boldsymbol{B}) \cap L^\infty(\boldsymbol{B})$ satisfying that $\int_{\boldsymbol{B}} (dd^c v)^n < +\infty$. Let a increasing sequence $\{\boldsymbol{B}_j\}$ of balls be such that $\boldsymbol{B}_j \subset\subset \boldsymbol{B}$ and $\bigcup_{j=1}^{+\infty} \boldsymbol{B}_j = \boldsymbol{B}$. Take a defining function ρ of $\boldsymbol{B}$, i.e., ρ is continuous plurisubharmonic function in $\boldsymbol{B}$ such that $\{\rho < -c\} \subset\subset \boldsymbol{B}$, for every $c > 0$ and $\rho = 0$ on $\partial\boldsymbol{B}$. Fix $j \geq 1$. Set $\mu_j := 1_{\boldsymbol{B}_j}\mu$. Since $\boldsymbol{B}_j \subset\subset \boldsymbol{B}$, then we choose a positive constant M_j such that $M_j\rho \leq v$ on $\overline{\boldsymbol{B}_j}$. We also take a decreasing sequence $\{w_j\}$ of continuous functions in neighborhood Ω of $\overline{\boldsymbol{B}}$ such that $w_j \searrow w$ in Ω.

Now, fix $j, k \geq 1$. By applying Theorem 4.2 and Corollary 3.4 in [10], we get that there exists a unique function $h_{w_k} \in PSH(\omega, \boldsymbol{B}) \cap L^\infty(\boldsymbol{B})$ such that $h_{w_k} = w_k$ on $\partial\boldsymbol{B}$ and $(\omega + dd^c h_{w_k})^n = 0$ in $\boldsymbol{B}$. Therefore we get the following estimates:

$$\mu_j \leq (dd^c \max(M_j\rho, v))^n$$

and

$$F(t,\cdot)d\mu_j \leq (dd^c \max(M_j\rho, v))^n \leq (\omega + dd^c \max(M_j\rho, v) + dd^c h_{w_k})^n,$$

$\forall t \in \mathbb{R}$,

in $\boldsymbol{B}$. By applying Lemma 2.3 for $F(t,z)$, w_k and μ_j, then there exists a unique function $u_{jk} \in PSH(\omega, \boldsymbol{B}) \cap L^\infty(\boldsymbol{B})$ such that $F(u_{jk},\cdot)d\mu_j = (\omega + dd^c u_{jk})^n$ and $u_{jk} = w_k$ on $\partial\boldsymbol{B}$. Since $w_k \geq w_{k+1}$ in $\partial\boldsymbol{B}$ and $F(u_{j(k+1)},\cdot)d\mu_j = (\omega + dd^c u_{j(k+1)})^n$ in $\boldsymbol{B}$ and $u_{j(k+1)} = w_{k+1}$ on $\partial\boldsymbol{B}$, by using Proposition 2.2 in [8] and the inequality (4) we obtain $u_{jk} \geq u_{j(k+1)} \geq w$ in $\boldsymbol{B}$.

On the other hand, since $\mu_j \leq \mu_{j+1}$ and $F(u_{(j+1)k},\cdot)d\mu_{j+1} = (\omega + dd^c u_{(j+1)k})^n$ and $u_{jk} = u_{(j+1)k} = w_k$ on $\partial\boldsymbol{B}$, by applying Proposition 2.2 in [8] and the inequality (4) we also get that $u_{jk} \geq u_{(j+1)k} \geq w$ in $\boldsymbol{B}$.

Set $u_j = \lim_{k\to\infty} u_{jk}$. Then $u_j \in PSH(\omega, \boldsymbol{B})$ and $u_j \geq w$ in $\boldsymbol{B}$. Since $u_{jk} \geq u_{j(k+1)} \geq u_{(j+1)(k+1)}$ and $u_j \leq u_{jk}, \forall k \geq 1$, then we get that $u_j \geq u_{j+1}$ in $\boldsymbol{B}$ and $\limsup_{z\to\zeta} u_j(z) \leq w(\zeta), \forall\zeta \in \partial\boldsymbol{B}$. Hence, by applying Corollary 3.4 in [10] and the estimate $\boldsymbol{w} \leq \boldsymbol{f}$ in X, we obain that $u_j \leq f$ in $\boldsymbol{B}$.

Now, we set $\widehat{w} = \lim_{j\to\infty} u_j$. Then $w \leq \widehat{w} \leq f$ and $\widehat{w} \in PSH(\omega, \boldsymbol{B})$. Moreover, we also get that $\limsup_{z\to\zeta} \widehat{w}(z) \leq w(\zeta), \forall\zeta \in \partial\boldsymbol{B}$.

Since u_j decreases pointwise to $\widehat{w}$, then we get that $(\omega + dd^c u_j)^n$ is weakly convergent to $(\omega + dd^c\widehat{w})^n$ as $j \to \infty$. On the other hand, by applying Lebesgue-dominated convergence theorem, we get that in each $\boldsymbol{B}_k$ for $k \geq 1$, $F(u_j,\cdot)d\mu_j$ is weakly convergent to $F(\widehat{w},\cdot)d\mu$ as j tends to ∞. It follows that $F(u_j,\cdot)d\mu_j$ weakly converges to $F(u,\cdot)d\mu$ in $\boldsymbol{B}$. Thus, we obtain that

$$(\omega + dd^c\widehat{w})^n = F(\widehat{w},\cdot)d\mu \qquad in \quad \boldsymbol{B}.$$

Set

$$\widetilde{w} = \begin{cases} \max(\widehat{w}, w) & on \quad \boldsymbol{B} \\ w & on \quad X\backslash\boldsymbol{B}. \end{cases}$$

Then $\widetilde{w} \in PSH(\omega, X), w \leq \widetilde{w} \leq f$. On the other hand, we also get that: $F(\widetilde{w},\cdot)d\mu \leq (\omega + dd^c\widetilde{w})^n$ in $X\backslash\boldsymbol{B}$ and $F(\widetilde{w},\cdot)d\mu = (\omega + dd^c\widetilde{w})^n$ in $\boldsymbol{B}$. By the definition of w, we get that $w = \widetilde{w}$ in X. Then we have that $F(w,\cdot)d\mu = (\omega + dd^c w)^n$ in $\boldsymbol{B}$. Since $\boldsymbol{B}$ is arbitrary, then we obtain that $F(w,\cdot)d\mu = (\omega + dd^c w)^n$ in X. The proof is complete.

3. CONCLUSION

In our paper, we proved a result (stated in Theorem 1.1) for solving the Monge-Ampère type equation $(\omega + dd^c u)^n = F(u,z)d\mu$ on a Hermitian manifold with boundary. This result is a slight extension of the results in [8]. There are still many open problems about solving these equations such as: study the Hölder continuity and the continuity of solutions; extend condition of $F(t,z)$ and μ.

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SỰ TỒN TẠI CỦA CÁC NGHIỆM BỊ CHẶN ĐỐI VỚI CÁC PHƯƠNG TRÌNH LOẠI MONGE-AMPÈRE PHỨC TRÊN CÁC ĐA TẠP HERMITIAN CÓ BIÊN

Tóm tắt: Trong bài báo này, chúng ta sẽ nghiên cứu và bàn luận về sự tồn tại của nghiệm bị chặn đối với phương trình loại Monge-Ampère phức $F(u,z)d\mu = (\mu + dd^c u)^n$ trên đa tạp Hermitian $(\overline{X},\omega)$ có biên khác rỗng, ở đây, u là một hàm chưa biết, μ là một độ đo Radon không âm trên X và $F(t,z)$ là một hàm $dt \times d\mu$ -đo được.

Từ khóa: các nghiệm bị chặn, các dưới nghiệm bị chặn, phương trình loại Monge-Ampère phức, các đa tạp Hermitian, các hàm ω -đa điều hòa dưới.