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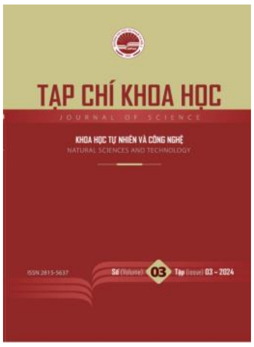

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On the second-order sufficient optimality condition in nonconvex multiobjective optimization problems

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Abstract

The study of second-order optimality conditions is one of the most important topics in optimization theory and attracting the attention and interest of many authors. In this paper, we introduce a novel solution concept called "essential local efficient solutions of second-order" for nonconvex constrained multiobjective optimization problems. We then show that the new solution concept is stronger than the quadratic growth condition and under a mild constraint qualification, these solution concepts are equivalent. By using the second subderivative, we derive a sufficient optimality condition for a feasible solution to become an essential local efficient solution of second-order for the considered problem. Examples are provided to illustrate the obtained results.

Keywords: Essential local efficient solutions of second-order, second subderivative, second-order sufficient optimality condition

1. Introduction

Second-order optimality conditions have long been recognized as an important tool in optimization theory and, in recent years, have been developed rapidly, see, for example [1]–[16]. It is well known that first-order optimality conditions are usually not sufficient for optimality except in the case of convex optimization problems. Second-order optimality conditions not only complement first-order ones in eliminating non-optimal solutions, but they also provide criteria for recognizing the optimality at a given feasible solution.

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In this paper, we will focus on second-order sufficient optimality conditions for the following constrained multiobjective optimization problem

$$\text{Min}_{\mathbb{R}^m_+} \left\{ f(x) : g(x) \in C \right\}, \tag{MP}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable mappings, and $C \subset \mathbb{R}^n$ is a nonempty and closed set. When $m = 1$, the above problem is called a mathematical program problem and is denoted by (P).

The study of second-order optimality conditions for (P), when C is convex, has been completely developed by Bonnas and Shapiro [1], Cominetti [2], Rockafellar and Wets [12], Mohammadi *et al*. [10], etc. More precisely, if C is convex polyhedral, second-order optimality conditions can be expressed in term of second derivative of the Lagrangian, see, for example [1], [12]. If C lacks the polyhedrality, then an additional term is needed to capture the curvature of C and there are various tools that can be utilized for such purpose, see [2], [10].

Recently, several important problem classes which can be reformulated in the form of problem (P) with non-convex C, such as, the mathematical program with complementarity constraints, the mathematical program with semi-definite cone complementarity constraints, etc. have attracted significant attention from the optimization community, see [17]–[20]. In these papers, the authors use the so-called lower generalized support function and the second subderivative to derive necessary and sufficient optimality conditions for (P) with C nonconvex. However, to the best of our knowledge, no papers have yet investigated second-order optimality conditions for multiobjective optimization problems of the form (MP). Motivated by the works reported in [11], [17], [18], in this paper, we introduce a new solution concept called "essential local efficient solutions of second-order" for the problem (MP) and study the sufficient optimality condition for the proposed solution.

We organize the paper as follows. Section 2 contains the preliminaries and auxiliary results. In Section 3, we present a second-order sufficient optimality condition for a feasible solution to be an essential local efficient solution of second-order to (MP). Section 4 provides some concluding remarks.

2. Preliminaries

Throughout this work we deal with the Euclidean space $\mathbb{R}^n$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. We denote by $B_r(x)$ the open ball centered at x with radius r. The set of all positive integer numbers is denoted by $\mathbb{N}$. Let Ω be a nonempty subset in $\mathbb{R}^n$. The *closure*, *interior*, *convex hull*, and *conic hull* of Ω are denoted, respectively, by $\text{cl}\,\Omega$, $\text{int}\,\Omega$, conv Ω, and cone$\,\Omega$. The distance $\text{dist}(x, \Omega)$ from a point $x \in \mathbb{R}^n$ to Ω is defined by

$$\text{dist}(x, \Omega) := \inf \left\{ \|y - x\| : y \in \Omega \right\} \quad \forall x \in \mathbb{R}^n.$$

The indicator function δ_Ω and the support function σ_Ω of Ω are defined, respectively, by

$$\sigma_\Omega(z^*) = \sup\left\{ \langle z^*, z \rangle : z \in \Omega \right\},$$

$$\delta_\Omega(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$, $z \in \Omega$, and $u \in \mathbb{R}^n$.

(i) The set *tangent/contingent cone* to Ω at z is defined by

$$T_\Omega(z) := \{u \in \mathbb{R}^n : \exists t_k \downarrow 0,\, u_k \to u \text{ with } z + t_k u_k \in \Omega \ \forall k \in \mathbb{N}\}.$$

(ii) The *second-order tangent set* to Ω at z with respect to the direction u is defined by

$$T^2_\Omega(z,u) := \left\{v \in \mathbb{R}^n : \exists t_k \downarrow 0, \exists v^k \to v, z + t_k u + \frac{1}{2} t_k^2 v^k \in \Omega,\ \forall k \in \mathbb{N}\right\}.$$

Remark 2.2. It is well-known that $T_\Omega(z)$ is a nonempty closed cone. For each $u \in \mathbb{R}^n$, the set $T^2_\Omega(z,u)$ is closed and $T^2_\Omega(z,u) = \varnothing$ if $u \notin T_\Omega(z)$. However, we see that the set $T^2_\Omega(z,0) = T_\Omega(z)$ is always nonempty. If Ω is convex, then we have

$$T_\Omega(z) = \text{cl}\,\{d : d = \beta(x - z),\ \ x \in \Omega, \beta \geq 0\},$$

and for each $u \in T_\Omega(z)$ one has

$$T^2_\Omega(z,u) \subset \text{cl cone[cone}(\Omega - z) - u].$$

Moreover, if Ω is a polyhedral convex set, then we have

$$T^2_\Omega(z,u) = T_{T_\Omega(z)}(u).$$

Definition 2.3. Let $w \in \mathbb{R}^n$. For $\delta,\ \rho > 0$,

$$V_{\delta,\rho}(w) := \{w' \in B_\delta(0) : \| \|w\| w' - \|w'\| w \| \leq \rho \|w'\| \|w\|\}$$

is called a *directional neighborhood of direction* w.

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$, $z \in \Omega$, and $w \in T_\Omega(z)$. The *proximal prenormal cone* $\mathrm{N}^p_\Omega(z,w)$ and the *proximal normal cone* $N^p_\Omega(z,w)$ to Ω at z in the direction w are defined, respectively, by

$$\mathrm{N}^p_\Omega(z,w) := \left\{z^* \in \mathbb{R}^n : \exists \delta, \rho, \gamma > 0 \text{ such that } \langle z^*, z' - z\rangle \leq \gamma \|z' - z\|^2 \ \forall z' \in \Omega \cap \left(z + V_{\delta,\rho}(w)\right)\right\},$$

$$N^p_\Omega(z,w) := \mathrm{N}^p_\Omega(z,w) \cap w^\perp.$$

If $w \notin T_\Omega(z)$, we define $N^p_\Omega(z,w) = \mathrm{N}^p_\Omega(z,w) = \varnothing$.

Definition 2.5. Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, \infty]$ be an extended real-valued function and $z \in \mathbb{R}^n$ such that $|\varphi(z)| < \infty$ and $z^* \in \mathbb{R}^n$.

(i) The *subderivative* of φ at z is defined by

$$\mathrm{d}\varphi(z)(w) := \liminf_{\substack{t \downarrow 0, \\ w' \to w}} \frac{\varphi(z + tw') - \varphi(z)}{t} \quad \forall w \in \mathbb{R}^n.$$

(ii) The *second subderivative* of φ at z for z^* is defined by

$$\mathrm{d}^2\varphi(z, z^*)(w) := \liminf_{\substack{t \downarrow 0 \\ w' \to w}} \frac{\varphi(z + tw') - \varphi(z) - t\langle z^*, w'\rangle}{\frac{1}{2}t^2} \quad \forall w \in \mathbb{R}^n.$$

Remark 2.6. (see [12]) (i) The second subderivative has the homogeneity property, i.e.,

$$\mathrm{d}^2(\alpha\varphi)(z, z^*)(w) = \alpha \mathrm{d}^2\varphi\left(z, \frac{z^*}{\alpha}\right)(w) \ \forall \alpha > 0,$$

and

$$\mathrm{d}\varphi(z)(w) > \langle z^*, w\rangle \ \Rightarrow \ \mathrm{d}^2\varphi(z, z^*)(w) = \infty,$$

$$\mathrm{d}\varphi(z)(w) < \langle z^*, w\rangle \ \Rightarrow \ \mathrm{d}^2\varphi(z, z^*)(w) = -\infty.$$

(ii) If φ is twice differentiable at z and $z^* = \nabla\varphi(z)$, then we have

$$\mathrm{d}^2\varphi(z, \nabla\varphi(z))(w) = w^T\nabla^2\varphi(z)w \ \forall w \in \mathbb{R}^n.$$

(iii) Let $\Omega \subset \mathbb{R}^n$, $z \in \Omega$, and $z^* \in \mathbb{R}^n$. Then, by definition, we have

$$\mathrm{d}^2\delta_\Omega(z; z^*)(w) = \liminf_{\substack{t\downarrow 0\\ w'\to w}} \frac{\delta_\Omega(z+tw') - \delta_\Omega(z) - t\langle z^*, w'\rangle}{\frac{1}{2}t^2} = \liminf_{\substack{t\downarrow 0,\, w'\to w\\ z+t w'\in\Omega}} \frac{-2\langle z^*, w'\rangle}{t}.$$

We now summarize some properties of the second subderivative that will be used in the next section.

Lemma 2.7. (see [17, Lemma 2.7]) *Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lower semicontinuous function and $z \in \mathbb{R}^n$ such that $|\varphi(z)| < \infty$ and $z^* \in \mathbb{R}^n$. Then there exist sequences $t_k \downarrow 0$* and *$w_k \to w$ such that*

$$\mathrm{d}\varphi(z)(w) = \lim_{k\to\infty} \frac{\varphi(z + t_k w_k) - \varphi(z)}{t_k},$$

$$\mathrm{d}^2\varphi(z, z^*)(w) = \lim_{k\to\infty} \frac{\varphi(z + t_k w_k) - \varphi(z) - t_k\langle z^*, w_k\rangle}{\frac{1}{2}t_k^2}.$$

Lemma 2.8. (see [18, Proposition 2.18]) *Let $\Omega \subset \mathbb{R}^n$, $z \in \Omega$, and $z^*, w \in \mathbb{R}^n$. The following statements hold:*

(i) *If $w \in T_\Omega(z)$ or $\langle z^*, w\rangle < 0$, then $\mathrm{d}^2\delta_\Omega(z; z^*)(w) = \infty$.*

(ii) *For $w \in T_\Omega(z)$, $\mathrm{d}^2\delta_\Omega(z; z^*)(w) > -\infty$ iff $z^* \in \mathrm{N}_\Omega^p(z, w)$.*

(iii) *If $\mathrm{d}^2\delta_\Omega(z; z^*)(w)$ is finite, then $z^* \in N_\Omega^p(z, w)$.*

(iv) *$\mathrm{d}^2\delta_\Omega(z; z^*)(w) \le -\sigma_{T_\Omega^2(z,w)}(z^*)$ iff $w \in T_\Omega(z)$ and $\langle z^*, w\rangle \ge 0$ or $T_\Omega^2(z, w) = \varnothing$.*

3. Second-order sufficient optimality conditions

Consider the following constrained optimization problem

$$\mathrm{Min}_{\mathbb{R}^m_+} \ \{f(x) : g(x) \in C\}, \qquad \text{(MP)}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable mappings with $f(x) := (f_1(x), \ldots, f_m(x))$, $g(x) := (g_1(x), \ldots, g_p(x))$, and $C \subset \mathbb{R}^n$ is a *nonempty and closed* set. The feasible set of (MP) is denoted by

$$S := g^{-1}(C) = \{x \in \mathbb{R}^n : g(x) \in C\}.$$

We *always* assume that S is nonempty. We say that u is a *critical direction* of problem (MP) at $\bar{x} \in S$ if

$$\begin{cases} \langle \nabla g(\bar{x}), u \rangle \in T_C(g(\bar{x})), \\ \langle \nabla f_i(\bar{x}), u \rangle \leq 0, \ \forall i \in I := \{1, \ldots, m\}. \end{cases}$$

The set of all critical directions of (MP) at $\bar{x} \in S$ is denoted by $K(\bar{x})$. We say that the set-valued mapping $x \rightrightarrows g(x) - C$ is *metrically subregular* at $(\bar{x}, 0)$ in direction $u \in \mathbb{R}^n$ if there exist $\kappa, \delta, \rho > 0$ such that

$$\operatorname{dist}(x, S) \leq \kappa \operatorname{dist}(g(x, C)) \quad \forall x \in \bar{x} + V_{\delta, \rho}(u).$$

The generalized Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p \to \mathbb{R}$ with respect to the problem (MP) is given as

$$L(x, \lambda, \mu) := \langle \lambda, f(x) \rangle + \langle \mu, g(x) \rangle.$$

We now introduce the concept of essential local efficient solutions of second-order for (MOP) inspired by the work of Penot [11].

Definition 3.1. Let $\bar{x} \in S.$ We say that:

(i) $\bar{x}$ is a *local efficient solution* of (MP) if there exists $\delta > 0$ such that there is no $x \in S \cap B_\delta(\bar{x})$ satisfying

$$f(x) \in f(\bar{x}) - \mathbb{R}^m_+ \setminus \{0\}.$$

(ii) $\bar{x}$ satisfies *the quadratic growth condition* if there exist two positive numbers $\beta > 0$ and $\delta > 0$ such that

$$\psi(x) := \max\{f_1(x) - f_1(\bar{x}), \ldots, f_m(x) - f(\bar{x})\} \geq \beta \|x - \bar{x}\|^2 \quad \forall x \in S \cap B_\delta(\bar{x}).$$

(iii) $\bar{x}$ is an *essential local efficient solution of second-order* for problem (MP) if there exist two positive numbers $\gamma > 0$ and $\delta > 0$ such that

$$\varphi(x) := \max\{f_1(x) - f_1(\bar{x}), \ldots, f_m(x) - f(\bar{x}), \operatorname{dist}(g(x), C)\} \geq \gamma \|x - \bar{x}\|^2 \quad \forall x \in B_\delta(\bar{x}).$$

The following result gives the relationships between above solution concepts.

Proposition 3.2. *Consider the following statements:*

(i) $\bar{x}$ *is a local efficient solution of (MP).*

(ii) $\bar{x}$ *satisfies the quadratic growth condition.*

(iii) $\bar{x}$ *is an essential local efficient solution of second-order for problem (MP).*

Then the implications (iii) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$(i) *always hold. Furthermore, if the mapping* $x \rightrightarrows g(x) - C$ *is metrically subregular at* $(\bar{x}, 0)$ *in every critical direction* $u \in K(\bar{x}) \setminus \{0\}$, *then the implication* (ii) $\Rightarrow$(iii) *is also valid*.

Proof. (iii) $\Rightarrow$(ii): The proof follows immediately from the definitions.

(ii) $\Rightarrow$(i): Suppose, for the sake of contradiction that $\bar{x}$ satisfies the quadratic growth condition but not is a local efficient solution of (MP). Then, by definition, there exist two positive numbers

$\beta > 0,\ \delta > 0,$ and some $x_0 \in S \cap B_\delta(\bar{x})$ such that

$$\max\{f_1(x) - f_1(\bar{x}), \ldots, f_m(x) - f(\bar{x})\} \geq \beta\|x - \bar{x}\|^2 \quad \forall x \in S \cap B_\delta(\bar{x}). \tag{1}$$

and

$$f_i(x_0) \leq f_i(\bar{x}) \ \forall i \in I \tag{2}$$

with at least one strict inequality. By (2), $x_0 \neq \bar{x}$. This and (1) imply that

$$\max\{f_1(x_0) - f_1(\bar{x}), \ldots, f_m(x_0) - f(\bar{x})\} \geq \gamma\|x - \bar{x}\|^2 > 0,$$

contrary to (2).

We now show that (ii) $\Rightarrow$ (iii) under the metric subregularity of the mapping $g(\cdot) - C$ at $(\bar{x}, 0)$. Suppose, for the sake of contradiction that $\bar{x}$ is not an essential local efficient solution of second-order for (P). Then for any $k \in \mathbb{N}$, there exists $x_k \in B_{1/k}(\bar{x})$ such that

$$\varphi(x_k) = \max\{f_1(x_k) - f_1(\bar{x}), \ldots, f_m(x_k) - f_m(\bar{x}), \operatorname{dist}(g(x_k), C)\} < \frac{1}{k}\|x_k - \bar{x}\|^2. \tag{3}$$

It is clear that $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$ and $x_k \to \bar{x}$. Hence,

$$\liminf_{k\to\infty} \frac{f_i(x_k) - f_i(\bar{x})}{\|x_k - \bar{x}\|^2} \leq 0 \quad \forall i \in I. \tag{4}$$

By passing to a subsequence we may assume that $\dfrac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to u \neq 0$ as $k \to \infty$. We first claim that $u \in K(\bar{x})$. Indeed, for each $k \in \mathbb{N}$, put $t_k := \| x_k - \bar{x} \|$ and $u_k := \dfrac{1}{t_k}(x_k - \bar{x})$. Then it follows from (3) that $\operatorname{dist}(g(x_k), C) < \dfrac{1}{k}t_k{}^2$. Hence, there exists $y_k \in C$ such that $\|g(x_k) - y_k\| < \dfrac{1}{k}t_k^2$. Put $r_k := \dfrac{y_k - g(x_k)}{t_k^2}$. Then we see that $\|r_k\| \to 0$ as $k \to \infty$ and $y_k = g(x_k) + t_k^2 r_k \in C$ for all $k \in \mathbb{N}$. Since $x_k = \bar{x} + t_k u_k$, by Taylor's expansion, we have

$$g(x_k) = g(\bar{x}) + t_k\langle \nabla g(\bar{x}), u_k\rangle + o(t_k).$$

Hence,

$$v_k := \frac{g(x_k) + t_k^2 r_k - g(\bar{x})}{t_k} = \frac{t_k\langle \nabla g(\bar{x}), u_k\rangle + o(t_k) + t_k^2 r_k}{t_k} = \langle \nabla g(\bar{x}), u_k\rangle + t_k r_k + \frac{o(t_k)}{t_k} \to \langle \nabla g(\bar{x}), u\rangle$$

as $k \to \infty$. Furthermore, $g(\bar{x}) + t_k v_k = g(x_k) + t_k^2 r_k = y_k \in C$ for all $k \in \mathbb{N}$. Hence, $\langle \nabla g(\bar{x}), u\rangle \in T_C(g(\bar{x}))$. We deduce again from (3) that $f_i(x_k) - f_i(\bar{x}) < \dfrac{1}{k}t_k^2$ for all $k \in \mathbb{N}$ and $i \in I$. This and the Taylor's expansion of f_i at $\bar{x},\ i \in I$, imply that $\langle \nabla f_i(\bar{x}), u\rangle \leq 0, \forall i \in I$, and hence, $u \in K(\bar{x})$, as required.

Now, by the assumption on the metric subregularity of the mapping $g(\cdot) - C$ at $(\bar{x}, 0)$ in the direction $u \in K(\bar{x})$, there exists $\ell > 0$ such that for all k large enough we can find some $\hat{x}_k \in S$ with

$$\|\hat{x}_k - x_k\| \le \ell \operatorname{dist}(g(x_k), C) < \frac{\ell}{k} t_k^2.$$

Clearly, $\hat{x}_k \to \bar{x}$ as $k \to \infty$ and $\|\hat{x}_k - x_k\| \le o(\|x_k - \bar{x}\|^2)$. Since $\bar{x}$ satisfies the quadratic growth condition, there exist two positive numbers $\beta > 0$ and $\delta > 0$ satisfying (1). Hence, for each k large enough, there exists $i_k \in I$ such that

$$f_{i_k}(\hat{x}_k) - f_{i_k}(\bar{x}) \ge \beta \|\hat{x}_k - \bar{x}\|^2. \tag{5}$$

Let I_k be the set of all indices $i_k \in I$ satisfying (5). Since $I_k \subset I$ for all k, without any loss of generality, we may assume that $I_k = \bar{I}$ is constant for all $k \in \mathbb{N}$ large enough. Fix $i \in \bar{I}$, then one has

$$f_i(\hat{x}_k) - f_i(\bar{x}) \ge \beta \|\hat{x}_k - \bar{x}\|^2$$

for all k large enough. Clearly, f_i is locally Lipschitz around $\bar{x}$ with some constant $l_i > 0$. Hence,

$$0 < \beta \le \liminf_{k\to\infty} \frac{f_i(\hat{x}_k) - f_i(\bar{x})}{\|\hat{x}_k - \bar{x}\|^2} \le \liminf_{k\to\infty} \frac{f_i(x_k) - f_i(\bar{x}) + l_i\|\hat{x}_k - x_k\|}{(\|x_k - \bar{x}\| - \|\hat{x}_k - x_k\|)^2}$$

$$= \liminf_{k\to\infty} \frac{f_i(x_k) - f_i(\bar{x}) + o(\|\hat{x}_k - x_k\|^2)}{\|\hat{x}_k - x_k\|^2 + o(\|\hat{x}_k - x_k\|^2)} = \liminf_{k\to\infty} \frac{f_i(x_k) - f_i(\bar{x})}{\|\hat{x}_k - x_k\|^2},$$

contrary to (4). The proof is complete. □

Remark 3.3. The converse of Proposition 3.2 is not true in general. For example, let $f: \mathbb{R} \to \mathbb{R}^2, x \mapsto (x^3, -x^3)$, $g: \mathbb{R} \to \mathbb{R}, x \to -x^2$, and $C = -\mathbb{R}_+$. Clearly, $S = \mathbb{R}$ and $\bar{x} = 0$ is an efficient solution of (MP). We claim that $\bar{x}$ is not an essential local efficient solution of second-order. Indeed, if otherwise, there exist $\gamma > 0$ and $\delta > 0$ such that

$$\max\{f_1(x) - f_1(\bar{x}), f_2(x) - f_2(\bar{x}), \operatorname{dist}(g(x), C)\} \ge \gamma \|x - \bar{x}\|^2 \quad \forall x \in B_\delta(\bar{x}),$$

or, equivalently, $\max\{x^3, -x^3\} \ge \gamma |x|^2 \;\; \forall x \in (-\delta, \delta)$. This implies that $|x| \ge \gamma \;\; \forall x \in (-\delta, \delta)$, a contradiction.

The following result gives a sufficient optimality condition for an an essential local efficient solution of second-order of problem (MP).

Theorem 3.4. *Let $\bar{x}$ be a feasible solution of (MP). Suppose that for every $u \in K(\bar{x})$, $\{0\}$ there exist $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}^p$ not both zero such that the following conditions hold:*

$$\nabla_x L(\bar{x}, \lambda, \mu) = 0, \tag{6}$$

$$\nabla_{xx}^2 L(\bar{x}, \lambda, \mu)(u, u) + \mathrm{d}^2 \delta_C(g(\bar{x}); \mu)(\nabla g(\bar{x}) u) > 0. \tag{7}$$

Then $\bar{x}$ is an essential local efficient solution of second-order of problem (MP).

Proof. The proof of the theorem follows some ideals of Benko et al. [18]. Suppose, for the sake of contradiction that $\bar{x}$ is not an essential local efficient solution of second-order of problem (MP). Then for each $k \in \mathbb{N}$, there exists $x_k \in B_{1/k}(\bar{x})$ such that

$$\varphi(x_k) = \max\left\{ f_1(x_k) - f_1(\bar{x}), \ldots, f_m(x_k) - f_m(\bar{x}), \operatorname{dist}(g(x_k), C) \right\} < \frac{1}{k}\|x_k - \bar{x}\|^2.$$

This implies that

$$f_i(x_k) - f_i(\bar{x}) \le \frac{1}{k}\|x_k - \bar{x}\|^2, \ i = 1, \ldots, m, \tag{8}$$

$$\operatorname{dist}(g(x_k), C) \le \frac{1}{k}\|x_k - \bar{x}\|^2. \tag{9}$$

Clearly, $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$ and $x_k \to \bar{x}$ as $k \to \infty$. For each $k \in \mathbb{N}$, put $t_k := \|x_k - \bar{x}\|$ and $u_k = t_k^{-1}(x_k - \bar{x})$. Since $\|u_k\| = 1$, by passing a subsequence if necessary we may assume that $u_k \to u$ with $\|u\| = 1$ as $k \to \infty$. It follows from (8) that

$$\lim_{k\to\infty} \frac{f_i(x_k) - f_i(\bar{x})}{t_k} = \lim_{k\to\infty} \frac{\langle \nabla f_i(\bar{x}), t_k u_k \rangle + o(t_k)}{t_k} = \langle \nabla f_i(\bar{x}), u \rangle \le 0 \tag{10}$$

and

$$\liminf_{k\to\infty} -\frac{f_i(\bar{x} + t_k u_k) - f_i(\bar{x})}{\frac{1}{2}t_k^2} \ge \lim_{k\to\infty} \frac{1}{k} = 0, \ i = 1, \ldots, m. \tag{11}$$

By (9) and the closedness of C, for each $k \in \mathbb{N}$, there exists $c_k \in C$ such that

$$\|c_k - g(x_k)\| = \operatorname{dist}(g(x_k), C) \le \frac{1}{k} t_k^2.$$

Put $r_k := \dfrac{c_k - g(x_k)}{t_k^2}$ and $v_k := \dfrac{g(x_k) - g(\bar{x}) + t_k^2 r_k}{t_k}$. Then $r_k \to 0$ as $k \to \infty$ and

$$g(\bar{x}) + t_k v_k = g(x_k) + t_k^2 r_k = c_k \in C \text{ for all } k \in \mathbb{N}.$$

Moreover, by the differentiability of g one has

$$\lim_{k\to\infty} v_k = \lim_{k\to\infty} \frac{g(x_k) - g(\bar{x}) + t_k^2 r_k}{t_k} = \lim_{k\to\infty} \frac{t_k \langle \nabla g(\bar{x}), u_k \rangle + o(t_k) + t_k^2 r_k}{t_k} = \langle \nabla g(\bar{x}), u \rangle.$$

Hence $\langle \nabla g(\bar{x}), u \rangle \in T_C(g(\bar{x}))$. This and (10) imply that $u \in K(\bar{x})$. By the assumption of the theorem, there exist $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}^p$ satisfying (6) and (7). By definition of the second subderivative and (11), we have

$$\mathrm{d}^2\delta_C(g(\bar{x}); \mu)(\nabla g(\bar{x})u) = \liminf_{\substack{t\downarrow 0 \\ w\to\nabla g(\bar{x})u}} \frac{\delta_C(g(\bar{x}) + tw) - \delta_C(g(\bar{x})) - t\langle \mu, w \rangle}{\frac{1}{2}t^2} =$$

$$= \liminf_{\substack{t\downarrow 0, w\to\nabla g(\bar{x})u \\ g(\bar{x}) + tw \in C}} \frac{-\langle \mu, w \rangle}{\frac{1}{2}t} \le \liminf_{k\to\infty} \frac{-\langle \mu, t_k v_k \rangle}{\frac{1}{2}t_k^2}$$

$$= \liminf_{k\to\infty} \frac{-\langle \mu, g(x_k) - g(\bar{x}) + t_k^2 r_k \rangle}{\frac{1}{2}t_k^2}$$

$$\le \liminf_{k\to\infty} -\frac{\langle \lambda, f(x_k) - f(\bar{x})\rangle}{\frac{1}{2}t_k^2} + \liminf_{k\to\infty} -\frac{\langle \mu, g(x_k) - g(\bar{x}) + t_k^2 r_k\rangle}{\frac{1}{2}t_k^2}$$

$$\le \liminf_{k\to\infty} -\frac{L(x_k, \lambda, \mu) - L(\bar{x}, \lambda, \mu)}{\frac{1}{2}t_k^2}$$

$$= \liminf_{k\to\infty} -\frac{\nabla_x L(\bar{x}, \lambda, \mu)(t_k u_k) + \frac{1}{2}\nabla^2_{xx} L(\bar{x}, \lambda, \mu)(t_k u_k, t_k u_k) + o(t_k^2)}{\frac{1}{2}t_k^2}$$

$$= -\nabla^2_{xx} L(\bar{x}, \lambda, \mu)(u, u),$$

contrary to (7). The proof is complete. □

The following result is a consequence of Theorem 3.4 and Proposition 3.2.

Corollary 3.5. *Let* $\bar{x}$ *be a feasible solution of (MP). Suppose that for every* $u \in K(\bar{x})$ *there exist* $\lambda \in \mathbb{R}^m_+$ *and* $\mu \in \mathbb{R}^p$ *not both zero such that conditions* (6) *and* (7) *hold. Then* $\bar{x}$ *satisfies the quadratic growth condition.*

We finish this section by presenting an example to illustrate Theorem 3.4.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}^2,\ x \mapsto (x^2, -x^2)$, $g : \mathbb{R} \to \mathbb{R},\ x \to -x^2$, and $C = -\mathbb{R}_+$. Then, $S = g^{-1}(C) = \mathbb{R}$. We now show that $\bar{x} = 0$ is an essential local efficient solution of second-order of problem (MP). It is easy to check that $K(\bar{x}) = \mathbb{R}$. Choose $\lambda = (1, 0)$ and $\mu = 0$. Then we have $\nabla_x L(\bar{x}, \lambda, \mu) = 0$ and

$$\nabla^2_{xx} L(\bar{x}, \lambda, \mu) + \mathrm{d}^2\delta_C(g(\bar{x}); \mu)(\nabla g(\bar{x})u) = \nabla^2_{xx} L(\bar{x}, \lambda, \mu) + \mathrm{d}^2\delta_C(0; 0)(0) = 2 > 0.$$

Hence, by Theorem 3.4, $\bar{x}$ is an essential local efficient solution of second-order of problem (MP).

4. Conclusions

In this paper, we have presented a second-order sufficient optimality condition for an essential local efficient solution of second-order to nonconvex multiobjective optimization problems with operator constraint. It is meaningful if we can establish a necessary optimality condition for this problem that has no-gap between the proposed sufficient optimality condition. We aim to investigate this problem in future work.

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