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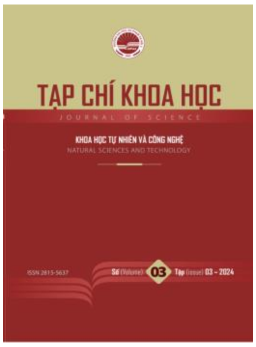

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On minimization of quadratic functions over closed convex sets in Hilbert spaces

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Abstract

Quadratic programming problems are of primary importance in various applications and arise as subproblems in many optimization algorithms. In this paper, we investigate quadratic programming problems in Hilbert spaces. By utilizing the Legendre property of quadratic forms and an asymptotically linear set with respect to a cone, we establish a sufficient condition for the existence of solutions to the considered problems through a Frank-Wolfe type theorem. The proposed condition is based on the special structure of Hilbert spaces, extending the applicability of quadratic programming methods. Finally, we provide a numerical example to illustrate the results obtained and demonstrate that existing approaches cannot be applied in certain cases.

Keywords: Quadratic program, Hilbert spaces, Legendre form, asymptotically linear, solution existence

1. Introduction

We consider the quadratic programming problems of the following form

$$\begin{cases} \min\ f(x) := \frac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle \\ \text{s.t.} \qquad x \in \Delta \subset \mathrm{H} \end{cases} \tag{QP}$$

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where H is a Hilbert space, $Q:\mathrm{H}\to\mathrm{H}$ is a continuous linear self-adjoint operator, $c\in\mathrm{H}$, Δ is a nonempty closed convex set, and $\langle\cdot,\cdot\rangle$ is the scalar product on H..

In 1956, Frank and Wolfe [1] proved the solution existence theorem for a (QP) problem. This result, called the Frank-Wolfe theorem, states that a quadratic function bounded from below over a nonempty polyhedral convex set in R^n attains its infimum there.

Theorem 1.1 (Frank-Wolfe Theorem)

Consider the following problem (QP_1)

$$\begin{cases}\min f(x):=\dfrac{1}{2}\langle Qx,x\rangle+\langle c,x\rangle \\ \text{s.t.} \qquad \Delta=\left\{x\in\mathrm{R}^n,\ x..b\right\}\end{cases} \qquad (\mathrm{QP}_1)$$

where $A\in\mathrm{R}^{m\times n}$, $c\in\mathrm{R}^m$ *and* $b\in\mathrm{R}^m$. *If* $\theta:=\inf\{f(x):x\in\Delta\}$ *is a finite real number then the problem has a solution.*

Many authors extended/generalized the Frank-Wolfe theorem to broader classes of functions and sets (see Blum and Oettli [2], Belousov [3], Belousov and Klatte [4], Bertsekas and Tseng [5], Semple [6], Schochetman, Smith and Tsui [7], Borwein [8], Martínez-Legaz, Noll and Sosa [9]).

Given a quadratic function and a polyhedral convex set, verifying whether the function is bounded from below on the set is a rather difficult task. In 1971, Eaves [10] gave a tool for dealing with the task.

Theorem 1.2 (The Eaves Theorem)

Problem (QP_1) *has solutions if and only if the following three conditions are satisfied:*

(i) Δ *is nonempty;*

(ii) *If* $v\in\mathrm{R}^n$ *and* $Av>0$ *then* $\langle Qv,v\rangle\geq 0$;

(iii) *If* $v\in\mathrm{R}^n$ *and* $x\in\mathrm{R}^n$ *are such that* $Av\geq 0$, $\langle Qv,v\rangle=0$ *and* $Ax\geq b$, *then* $\langle Qx+c,v\rangle\geq 0$.

In 1999, Luo and Zhang [11] proved the important result.

Theorem 1.3

Consider the following problem (QP_2)

$$\begin{cases}\min f(x):=\dfrac{1}{2}\langle Qx,x\rangle+\langle c,x\rangle \\ \text{s.t.} \quad \Delta=\left\{x\in\mathrm{R}^n:\dfrac{1}{2}\langle Q_ix,x\rangle+\langle c_i,x\rangle+\alpha_i,,\ 0\right\},\ i=1,2,...m\end{cases} \qquad (\mathrm{QP}_2)$$

with each $Q_i\in\mathrm{R}^{n\times n}$ *being positive semidefinite,* $c_i\in\mathrm{R}^n$ *and* $\alpha_i\in\mathrm{R}$.. *Suppose that* Q_1 *is positive semidefinite and* $Q_i=0$ *for* $i=2,3,\ldots,m$. *Then, if the objective function* $f(x)$ *is bounded over* Δ, *then the infimum of* (QP_2) *is attained.*

In 2000, by using the Legendre property of the quadratic form in the objective function, Bonnans and Shapiro [12] characterized the solution existence of the problem (QP) with Δ being a polyhedral set.

Theorem 1.4

Consider the problem (QP_3)

$$\begin{cases} \min f(x) := \dfrac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle \\ \text{s.t.} \quad \Delta = \left\{x \in \mathrm{H} : \langle c_i, x\rangle \leq b_i\right\}, \quad i = 1, 2, \ldots, m. \end{cases} \qquad (\mathrm{QP}_3)$$

Suppose that the quadratic form Q *is a Legendre form. Then the following conditions are equivalent:*

(i) Problem (QP_3) *has an optimal solution,*

(ii) The optimal value of (QP_3) *is finite.*

Consider the quadratic programming problem (QP). Denote

$$I = \{1, \ldots, m\}, \quad I_0 = \{i \in I \mid Q_i = 0\}, \quad I_1 = \{i \in I \mid Q_i \neq 0\} = I \setminus I_0.$$

Recently, by using the Legendre property of the quadratic form in the objective function and Condition (A), Dong and Tam [13] presented the following.

Theorem 1.5

Consider the problem (QP_4)

$$\begin{cases} \min f(x) := \dfrac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle \\ \text{s.t.} \quad \Delta = \left\{x \in \mathrm{H} : \dfrac{1}{2}\langle Q_i x, x\rangle + \langle c_i, x\rangle + \alpha_i \leq 0\right\}, \quad i = 1, 2, \ldots, m, \end{cases}, \qquad (\mathrm{QP}_4)$$

where H *is a Hilbert space,* $\langle Qx, x\rangle$ *is a Legendre form,* Q_i *is a positive semidefinite continuous linear self-adjoint operator on* H *,* $c, c_i \in \mathrm{H}$ *, and* α_i *are real number,* $i = 1, 2, \ldots, m$*. Suppose that f is bounded from below over nonempty* Δ *and the following condition:*

Condition A: *If* $I_1 \neq \varnothing$*, then*

$$\left(v \in 0^+\Delta, \ \langle Qv, v\rangle = 0\right) \Rightarrow \langle c_i, v\rangle = 0 \ \forall i \in I_1$$

is satisfied. Then, problem (QP_4) *has a solution.*

The purpose of this paper is to extend the results on solution existence for (QP) problem in Euclidean spaces to Hilbert spaces. The idea of using the Legendre property of the quadratic form in the objective function and weakly asymptotically linear in proving the main result.

2. Preliminaries

Let H be a real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\|.\|$.

* A sequence $\{x^k\}$ in H is said to be converge weakly to x^0, the notation $x^k \rightharpoonup x^0$, or $x^k \xrightarrow{w} x^0$, if $\langle x^k, a\rangle \to x^0$, a for each $a \in \mathrm{H}$.

* A sequence $\{x^k\}$ in H is said to be converge strongly to x^0, the notation $x^k \to x^0$, if $\|x^k - x^0\| \to 0$.

* A function $q:\mathrm{H} \to \mathbb{R}$ is said to be quadratic form on H if there exists a bilinear symmetric function $B(\cdot,\cdot)$ on H such that $q(x) = B(x,x)$.

In this paper, we will only consider the continuous quadratic forms By Riesz Theorem [14, Theorem 2.34] it admits the following representation $q(x) = \langle Qx, x\rangle$, where $Q:\mathrm{H} \to \mathrm{H}$ is a continuous linear self-adjoint operator.

The operator $Q:\mathrm{H} \to \mathrm{H}$ is said to be positive semidefinite (positive) if the quadratic from $q(x) = \langle Qx, x\rangle$ is nonnegative (positive, respectively).

A function $f(x) = \frac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle$, where $Q:\mathrm{H} \to \mathrm{H}$ is a continuous symmetric linear operator and $c \in \mathrm{H}$, is called quadratic.

Definition 2.1 (see [12, p.193] or [15])

A quadratic form $q:\mathrm{H} \to \mathrm{R}$ is said to be a Legendre form if

(i) it is weakly lower semicontinuous, and

(ii) $x_k \to x_0$ whenever $x_k \rightharpoonup x_0$ and $q(x_k) \to q(x_0)$.

It is clear that in the case where H is of finite dimension, any quadratic form $q(x)$ on H is a Legendre form.

A quadratic function $f(x) = \frac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle$ is said to be a Legendre function if $\langle Qx, x\rangle$ is a Legendre form.

Definition 2.2 (see [16])

Recession cone of a nonempty closed convex set $\Delta \subset \mathrm{H}$ is defined by

$$0^+\Delta = \{v \in \mathrm{H} \mid x + tv \in \Delta, \forall x \in \Delta,\ \forall t \geq 0\}.$$

Lemma 2.1 (see [13] and [16])

If Δ is nonempty, then

$$0^+\Delta = \{v \in \mathrm{H} \mid\ Q_i v = 0,\ \langle c_i, v\rangle \leq 0, \forall i = 1,\ldots,m\}.$$

The following property of $0^+\Delta$ will be used many times in our paper.

Lemma 2.2 (see [16])

Let $\Delta \subset \mathrm{H}$ be a nonempty closed convex set. If $\{x^k\} \subset \Delta, x^k \to +\infty$ and $\frac{x^k}{\|x^k\|} \rightharpoonup v$ as $k \to \infty$, then $v \in 0^+\Delta$.

Definition 2.3 (see [17, Definition 2.3.1] for the case $\mathrm{H} = \mathrm{R}^n$)

A nonempty and closed set C of H is said to be an *asymptotically linear* set if for each $\rho > 0$ and for each sequence $\{x^k\}$ satisfying

$$\{x^k\} \subset C, x^k \to +\infty, \frac{x^k}{\|x^k\|} \rightharpoonup v \text{ as } k \to \infty,$$

there exists $k_0 \in N$ such that $x^k - \rho v \in C \ \ \forall k \geq k_0$.

Remark 2.1

Every polyhedral set is asymptotically linear (see after [17], Definition 2.3.2] for the case $\mathrm{H} = \mathrm{R}^n$), but the converse is not true (see, for instance, [18]). Under the assumption that the constraint set is asymptotically linear, the existence of solution for quadratic programming and quadratic fractional programming has been studied in [14] and [19].

The concept of asymptotically linear sets with respect to a cone was introduced by Nghi [18].

Definition 2.4 (see [18] for the case $\mathrm{H} = \mathrm{R}^n$)

Let $C \subset \mathrm{H}$ be a nonempty closed and convex set and let $K \subset 0^+ C$ be a cone. Then, C is said to be asymptotically linear with respect to K if for each sequence $\{x^k\} \subset C$ satisfying $x^k \to +\infty$ and $\frac{x^k}{\|x^k\|} \xrightarrow{w} \bar{v} \in K$, there exists $\delta > 0$ such that $x^k - t\bar{v} \in \Delta$, for every $t \in (0, \delta)$ for every k large enough.

The class of asymptotically linear sets with respect to a cone K contains, in particular. The class of asymptotically linear sets. A set C is asymptotically linear if and only if C is asymptotically linear with respect to any $K \subset 0^+ C$.

We have the following lemma.

Lemma 2.3

Consider (QP) with $\Delta = \left\{ x \in \mathrm{H} : \frac{1}{2}\langle Q_i x, x\rangle + \langle c_i, x\rangle + \alpha_i \leq 0 \right\}, i = 1, 2, \ldots, m$, *where* Q_i *is a positive semidefinite continuous linear self-adjoint operator on* $\mathrm{H}, c_i \in \mathrm{H}$ *and* α_i *are real number. Let* $K = 0^+\Delta \cap \{ v \in \mathrm{H} \mid \langle Qv, v\rangle = 0 \}$. *If Condition (A) is satisfied then* Δ *is an asymptotically linear set with respect to* K.

Proof. By a similar argument as in [20, Theorem 2.1], we obtain the desired conclusion.

3. Main result

We now consider the quadratic programming problems of the following form

$$\begin{cases} \min f(x) := \frac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle \\ \text{s.t.} \qquad x \in \Delta \subset \mathrm{H} \end{cases} \tag{QP}$$

where H is a Hilbert space, $Q:\mathrm{H}\to\mathrm{H}$ is a continuous linear self-adjoint operator, $c\in\mathrm{H}$, Δ is a nonempty closed convex set. Let $K=0^{+}\Delta\cap\{v\in\mathrm{H}\mid\langle Qv,v\rangle=0\}$. We propose sufficient conditions for the solution existence of the problem (QP) through a Frank-Wolfe type theorem as follows.

Theorem 3.1

Suppose that $\langle Qx,x\rangle$ is a Legendre form, Δ is an asymptotically linear set with respect to K, and f is bounded from below over nonempty Δ. Then, the problem (QP) has a solution.

Proof. Let $f^{*}=\inf\{f(x)\mid x\in\Delta\}$. For each positive integer k, let $S_k=\left\{x\in\Delta\mid f(x)\le f^{*}+\frac{1}{k}\right\}$. By the continuity of f and $f^{*}>-\infty$, S_k is nonempty and closed. We show that S_k has an element of minimal norm. Let $y_0=\inf\{\|y\|,y\in S_k\}$.

There exists a sequence $\{y^k\subset S_k\}$ such that $y_0=\lim\limits_{k\to\infty}\|y^k\|$. Since $\{y^k\}$ is bounded, it has a weakly convergent subsequence, we can assume that $y^k\xrightarrow{w}\bar{y}$ as $k\to\infty$. Since Δ is closed convex set, by Mazur's theorem, Δ is weakly closed. Hence $\bar{y}\in\Delta$. Since $\langle Qy,y\rangle$ is weakly lower semicontinuous, we have

$$\langle Q\bar{y},\bar{y}\rangle\le\lim_{k\to\infty}\langle Qy^k,y^k\rangle.$$

From this, it follows that

$$f(\bar{y})=\frac{1}{2}\langle Q\bar{y},\bar{y}\rangle+\langle c,\bar{y}\rangle\le\lim_{k\to\infty}\left(\frac{1}{2}\langle Qy^k,y^k\rangle+\langle c,y^k\rangle\right).$$

$$\le f^{*}.$$

Combining this with $\bar{y}\in\Delta$ we have that $\bar{y}\in S_k$. Hence $y_0=\|\bar{y}\|$ and $\bar{y}$ is an element of the minimal norm of S_k. Consider the sequence $\{x^k\}$ in S_k and we prove that $\{x^k\}$ is bounded. Suppose that $\{x^k\}$ is unbounded. Without loss of generality we may assume that $\|x^k\|\to\infty$ as $k\to\infty$ and $\|x^k\|\ne0$ for all *k*. Let $v^k:=\frac{x^k}{\|x^k\|}$, we have $\|v^k\|=1$. Since $\{v^k\}$ is bounded, it has a weakly convergent subsequence, we may assume $v^k\xrightarrow{w}\bar{v}$. Because $\{x^k\}\in S_k$, we have

$$f(x_k)=\frac{1}{2}\langle Qx^k,x^k\rangle+\langle c,x^k\rangle\le f^{*}+\frac{1}{k}. \tag{1}$$

We prove the following

$$\langle Q\bar{v},\bar{v}\rangle=0 \text{ and } \langle Qx^k+c,\bar{v}\rangle\ge0, \text{ for each } k \tag{2}$$

By multiplying both sides of (1) by $\frac{1}{\|x^k\|^2}$ and letting $k\to\infty$ and using the fact that $\langle Qx,x\rangle$ is weakly lower semicontinuity, we have

$$\frac{1}{2}\left\langle Q\bar{v},\bar{v}\right\rangle \le \liminf_{k\to\infty}\frac{1}{2}\left\langle Qv^k, v^k\right\rangle \le 0. \tag{3}$$

If either $\left\langle Q\bar{v},\bar{v}\right\rangle < 0$ or $\left\langle Q\bar{v},\bar{v}\right\rangle = 0$ and $\left\langle Qx^k + c,\bar{v}\right\rangle < 0$ then

$$f\left(x^k + t\bar{v}\right) = f\left(x^k\right) + \frac{t^2}{2}\left\langle Q\bar{v},\bar{v}\right\rangle + t\left\langle Qx^k + c,\bar{v}\right\rangle \to -\infty \text{ as } t \to +\infty,$$

which contradicts the fact that f is bounded from below over Δ. Hence, we have $\left\langle Q\bar{v},\bar{v}\right\rangle = 0$ and $\left\langle Qx^k + c,\bar{v}\right\rangle \ge 0$.

We next prove that

$$\|\bar{v}\| \neq 0. \tag{4}$$

Multiplying both sides of the inequality in (1) by $\frac{1}{\|x^k\|^2}$ and let $k \to \infty$, we have

$$\limsup_{k\to\infty}\frac{1}{2}\left\langle Qv^k, v^k\right\rangle \le 0. \tag{5}$$

Combining (3), (5) and $\left\langle Q\bar{v},\bar{v}\right\rangle = 0$, we conclude that

$$\lim_{k\to\infty}\left\langle Qv^k, v^k\right\rangle = \left\langle Q\bar{v},\bar{v}\right\rangle. \tag{6}$$

Since $\left\langle Qx, x\right\rangle$ is a Legendre form, so that $v^k \xrightarrow{w} \bar{v}$ and (6), we have $v^k \to \bar{v}$. By the fact that $\|v^k\| = 1$ for all k, we obtain that $\|\bar{v}\| = 1$ so $\bar{v} \neq 0$. Since Δ is an asymptotically linear set, there exists $k_0 \in \mathrm{N}$ such that

$$x^k - t\bar{v} \in \Delta \text{ for all } k \ge k_0 \text{ and for all } t > 0 \text{ small enough.} \tag{7}$$

By (2), we have

$$\begin{aligned} f\left(x^k - t\bar{v}\right) &= \frac{1}{2}\left\langle Q\left(x^k - t\bar{v}\right), x^k - t\bar{v}\right\rangle + \left\langle c, x^k - t\bar{v}\right\rangle \\ &= \frac{1}{2} f\left(x^k\right) + \frac{t^2}{2}\left\langle Q\bar{v},\bar{v}\right\rangle - t\left\langle Qx^k + c,\bar{v}\right\rangle \\ &\le f\left(x^k\right). \end{aligned} \tag{8}$$

Combining (7) and (8), we have

$$x^k - t\bar{v} \in S_k,\ \forall k \ge k_0. \tag{9}$$

By (4) and $\frac{x^k}{\|x^k\|} = v^k, v^k \xrightarrow{w} \bar{v}$, we have

$$\lim_{k\to\infty}\frac{1}{\left\|x^k\right\|}\left\langle x^k,\overline{v}\right\rangle=\lim_{k\to\infty}\left\langle\frac{x^k}{\left\|x^k\right\|},\overline{v}\right\rangle=\left\langle\overline{v},\overline{v}\right\rangle=\overline{v}^2>0.$$

Hence, there exists $k_1\ge k_0$ such that

$$\left\langle x^k,\overline{v}\right\rangle>0,\ \forall k\ge k_1. \tag{10}$$

Let $\alpha=\inf\{\left\langle x^k,\overline{v}\right\rangle;\ k\ge k_1\}$. By (10), we have $\alpha\ge 0$.

We show that $\alpha>0$. If otherwise then there exists $\left\{k^m\right\}$ such that $\lim_{k^m\to\infty}\left\langle x^{k^m},\overline{v}\right\rangle=0.$ From this it follows that $\lim_{k^m\to\infty}\frac{1}{\left\|x^{k^m}\right\|}\left\langle x^{k^m},\overline{v}\right\rangle=0.$

On the other hand, $\lim_{k^m\to\infty}\frac{1}{\left\|x^{k^m}\right\|}\left\langle x^{k^m},\overline{v}\right\rangle=\left\langle\overline{v},\overline{v}\right\rangle.$ Hence $\overline{v}=0$, a contradiction (4). Thus $\alpha>0$.

We have

$$\left\|x^k-t\overline{v}\right\|^2=\left\|x^k\right\|^2-2t\left\langle x^k,\overline{v}\right\rangle+t^2\left\|\overline{v}\right\|^2<\left\|x^k\right\|^2,\forall t\in\left(0,\alpha\right). \tag{11}$$

Combining (9) and (11), we have $x^k-t\overline{v}\in S_k$ and $\left\|x^k-t\overline{v}\right\|<\left\|x^k\right\|,\forall k\ge k_1,\forall t\in\left(0,\alpha\right).$ This contradicts the fact that x^k is the element of the minimal norm in S_K. Hence, we have $\left\{x^k\right\}$ is bounded.

Since $\left\{x^k\right\}$ is bounded, it has a weakly convergent subsequence. We can assume that $x^k\xrightarrow{w}\overline{x}$. Since $x^k\in$ for all k and Δ is weakly closed, we have $\overline{x}\in\Delta.$ On the other hand, $\left\langle Qx,x\right\rangle$ is a Legendre form, and it therefore weakly lower semicontinuous, we have

$$\frac{1}{2}\left\langle Q\overline{x},\overline{x}\right\rangle\le\liminf_{k\to\infty}\frac{1}{2}\left\langle Qx^k,x^k\right\rangle.$$

Hence by (1), we have

$$f\left(\overline{x}\right)=\frac{1}{2}\left\langle Q\overline{x},\overline{x}\right\rangle+\left\langle c,\overline{x}\right\rangle\le\liminf_{k\to\infty}\left(\frac{1}{2}\left\langle Qx^k,x^k\right\rangle+\left\langle c,x^k\right\rangle\right)\le\liminf_{k\to\infty}\left(f^*+\frac{1}{k}\right)=f^*.$$

It follows that $\overline{x}$ is a solution of (QP). The proof is complete.

The following example is constructed to show that there is a (QP) whose constraint set is an asymptotically linear, but condition (A) applied for (QP) is not satisfied.

Example 3.1

Consider the programming problem (QP), where $\mathrm{H}=\mathrm{R}^2$

$$Q=\begin{bmatrix}-1&0\\0&0\end{bmatrix},\ Q_1=\begin{bmatrix}0&0\\0&0\end{bmatrix},\ Q_2=\begin{bmatrix}0&0\\0&0\end{bmatrix},\ Q_3=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ Q_4=\begin{bmatrix}1&0\\0&0\end{bmatrix},$$

$$c=\left(0,0\right),\ c_1=\left(-1,0\right),\ c_2=\left(1,0\right),\ c_3=\left(0,-1\right),\ c_4=\left(-2,-1\right),$$

$$\alpha_1=\alpha_2=-1,\alpha_3=0,\alpha_4=2.$$

We can rewrite the problem as follows:

$$\begin{cases} \min f(x) := -x_1^2 \\ \text{s.t.} \quad \Delta = \{x = (x_1; x_2) \in \mathrm{R}^2 \mid -x_1 - 1 \leq 0, x_1 - 1 \leq 0, \\ x_1^2 - x_2 \leq 0, x_1^2 - 2x_1 - x_2 + 2 \leq 0\}. \end{cases}$$

We have $0^+\Delta = \{v = (v_1, v_2) \in \mathrm{R}^2 : v_1 = 0, v_2 \geq 0\}$.

Let $\bar{v} = (0,1) \in 0^+\Delta$, we have $\langle Q\bar{v}, \bar{v} \rangle = 0$ but $\langle c_3, \bar{v} \rangle = -1 < 0$. Hence, condition (A) is not satisfied.

We next show that Δ is an asymptotically linear set. Indeed, suppose with a sequence $\{x^k\} \subset \Delta$ such that $\|x^k\| \to +\infty$, $\dfrac{x^k}{\|x^k\|} \to \bar{v} = (0,1) \in 0^+\Delta$. Let $x^k = (x_1^k, x_2^k) \in \Delta$ satisfy $|x_1^k| \leq 1$, $x_2^k \geq 0$ and $x_2^k \to +\infty$. Let $\delta > 0$, then there exists a number $k_0 \in \mathrm{N}$ such that $\forall k \geq k_0$ and $x_2^k - \delta \geq 5$. We have $\forall t \in (0, \delta) : x^k - t\bar{v} = (x_1^k, x_2^k - t)$. Since $(x_1^k)^2 \leq 1$ and $x_2^k - t \geq x_2^k - \delta \geq 5$, we have $(x_1^k)^2 - (x_2^k - t) < 0$. We obtain that $(x_1^k)^2 - 2x_1^k - (x_2^k - t) + 2 \leq 5 - (x_2^k - t) \leq 0$. So $x^k - t\bar{v} \in \Delta$ with $\forall k \geq k_0$. Thus, Δ is an asymptotically linear set.

Since $f(x) := -x_1^2 \geq -1$ for all $x \in \Delta$, by Theorem 3.1, we obtain that the problem in this example has a solution.

4. Conclusion

In this paper, we propose a sufficient condition for the solution existence of constrained quadratic programming in Hilbert spaces (Theorem 3.1). A numerical example is presented to illustrate the obtained result.

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