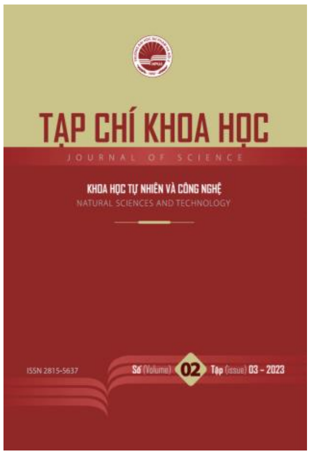
HPU2. Nat. Sci. Tech. Vol 02, issue 03 (2023), 13-18

HPU2 Journal of Sciences:
Natural Sciences and Technology

journal homepage: https://sj.hpu2.edu.vn



Article type: Research article

Applications of the Z transformation in calculating sum of a series

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Abstract

The Z transformation has many applications in Mathematics, especially in solving difference equations to help handle discrete data models. This article provides one more application of the Z transformation, which is the summation of a series. The author through the research method of theory development from some properties of the Z transformation to obtain the result which is a theorem about the formula for the sum of a series, with the proof attached. From there, apply the theorem together with the Z transformation to calculate the sum of some series in specific problems. Thus, the problem of calculating the sum of the series has one more method of solving, as well as expanding the application of the Z transformation in the field of Mathematics, especially analysis.

Keywords: The Z transformation, the inverse Z transformation, series, summary of series

1. Introduction

The Z transformation is useful tool in handling discrete data models, widely used in the fields of applied mathematics, digital signal processing, control theory and economics [1,2,3,4,5]. These discrete models are solved by difference equations similar to the continuous models solved by differential equations. The Z transform plays an important role in solving difference equations in the same way that the Laplace transform plays an important role in solving differential equations. In this article, the author mentions another application of the Z transformation to calculate series sums. The author has researched and developed the theory to obtain a way to calculate series sums through the Z

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https://doi.org/10.56764/hpu2.jos.2023.2.3.13-18

Received date: 13-10-2023 ; Revised date: 24-10-2023 ; Accepted date: 06-12-2023

transformation.

Below, the author will describe in detail how to approach the research problem, starting from introducing the concept and some related properties of the Z transformation, accompanied by illustrative examples. Then the author will give a way to calculate the series sum using the Z transformation by a theorem and some examples.

2. Research methods

The author uses the research method of theory development. First of all, we approach the definition of the Z transformation.

Definition 1. Let T be a fixed positive number (can be taken $T=1$). Suppose $f(t)$ identifies with $t\geq 0$ and t gets the value at $nT; n=0,1,2,...$. The Z transformation of function $f(t)$, or sequence $\{f(nT)\}$, is a complex variable function z determined by formula

$$Z\{f(nT)\}=F(z)=\sum_{n=0}^{\infty} f(nT)z^{-n},$$

where $|z|>R=\dfrac{1}{\rho}$, ρ is the radius of convergence of the series.

Example 1. Let $f(nT)=a^{nT}$. Then

$$Z\{a^{nT}\}=\sum_{n=0}^{\infty} a^{nT}z^{-n}=\sum_{n=0}^{\infty}\left(\frac{a^T}{z}\right)^n$$

$$=\frac{z}{z-a^T};|z|>a^T. \tag{1}$$

Example 2. The Z transformation of $f(nT)=\dfrac{1}{n!T!}$ is

$$Z\left\{\frac{1}{n!T!}\right\}=\sum_{n=0}^{\infty}\frac{1}{n!T!}z^{-n}$$

$$=\frac{1}{T!}\exp\left(\frac{1}{z}\right);\forall z \tag{2}$$

Example 3. With $f(nT)=\cos nT\omega$ we have

$$Z\{\cos nT\omega\}=\sum_{n=0}^{\infty}\frac{e^{inT\omega}+e^{-inT\omega}}{2}z^{-n}$$

$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}\left(e^{iT\omega}z^{-1}\right)^n+\sum_{n=0}^{\infty}\left(e^{-iT\omega}z^{-1}\right)^n\right]$$

$$=\frac{1}{2}\left(\frac{1}{1-e^{iT\omega}z^{-1}}+\frac{1}{1-e^{-iT\omega}z^{-1}}\right)$$

$$=\frac{z(z-\cos T\omega)}{z^2-2z\cos T\omega+1};|z|>1.$$

The Z transformation has many properties. Here, the author only states some necessary related properties:

i. Proportional properties

$$Z\left\{a^{nT}f(nT)\right\}=F(a^{-T}z). \tag{3}$$

ii. Multiplication

$$Z\left\{nf(nT)\right\}=-z\frac{dF(z)}{dz},$$

$$Z\left\{nTf(nT)\right\}=-Tz\frac{dF(z)}{dz}.$$

iii. Division

$$Z\left\{\frac{f(nT)}{n+m}\right\}=z^m\int_z^\infty\frac{F(z)}{z^{m+1}}dz;m\geq 0. \tag{4}$$

iv. Initial value theorem

Suppose $Z\left\{f(nT)\right\}=F(z)$. Then

$$f(0)=\lim_{z\to\infty}F(z).$$

v. Final value theorem

Suppose $Z\left\{f(nT)\right\}=F(z)$. Then

$$\lim_{n\to\infty}f(nT)=\lim_{z\to 1}[(z-1)F(z)], \tag{5}$$

where, the limits are assumed to exist.

vi. Inverse Z transformation

$$Z^{-1}\left\{F(z)\right\}=f(nT)=\frac{1}{2\pi i}\oint_C F(z)z^{n-1}dz,$$

where C is the closed circuit surrounding the origin and outside the circle $|z|=R$.

3. Results and discussion

Theorem 1. Suppose $Z\left\{f(nT)\right\}=F(z)$. Then

(i) $\sum_{k=0}^{n}f(k)=Z^{-1}\left\{\frac{z}{z-1}F(z)\right\},$

(ii) $\sum_{k=0}^{\infty}f(k)=\lim_{z\to 1}F(z)=F(1).$

Prove.

Set $g(n)=\sum_{k=0}^{n}f(k)$. Then

$$g(n)=f(n)-g(n-1).$$

Take the Z transformation both sides of the above equation, we get

$$G(z) = F(z) + z^{-1}G(z),$$

$$\Rightarrow G(z) = \frac{z}{z-1}F(z).$$

Therefore

$$g(n) = Z^{-1}\{G(z)\} = Z^{-1}\left\{\frac{z}{z-1}F(z)\right\}.$$

So

$$\sum_{k=0}^{n} f(k) = Z^{-1}\left\{\frac{z}{z-1}F(z)\right\}.$$

According to (5) we have $\lim_{n\to\infty} g(n) = \lim_{z\to 1}\left[(z-1)G(z)\right]$, or

$$\lim_{n\to\infty}\sum_{k=0}^{n} f(k) = \lim_{z\to 1}\left[(z-1)\frac{z}{z-1}F(z)\right] = F(1).$$

So, $\sum_{k=0}^{\infty} f(k) = F(1).$

We will apply the above theorem to some specific problems below.

Problem 1. Use the Z transformation to calculate the sum of the series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

According to (3) we have

$$Z\{x^n f(n)\} = F\left(\frac{z}{x}\right).$$

Set $f(n) = \frac{1}{n!}$. From result (2) we have $F(z) = Z\{f(n)\} = \exp\left(\frac{1}{z}\right)$. From there it can be deduced

$$Z\left\{\frac{x^n}{n!}\right\} = \exp\left(\frac{x}{z}\right).$$

Using Theorem 1 (ii) we get

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x.$$

Problem 2. Use the Z transformation to show that

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^n+1}{n+1} = \log(1+x).$$

Using (1) we have

$$Z\{x^{n+1}\} = \frac{xz}{z-x}.$$

According to (4) we get

$$Z\left\{\frac{x^{n+1}}{n+1}\right\}=z\int_z^\infty \frac{xz}{z-x}\cdot\frac{dz}{z^2}=xz\int_z^\infty \frac{dz}{z(z-x)}$$

$$=xz\left[\frac{1}{x}\log\left(\frac{z-x}{z}\right)\right]_z^\infty=-z\log\left(\frac{z-x}{z}\right).$$

Replaced x by $-x$,

$$Z\left\{(-1)^n\frac{x^{n+1}}{n+1}\right\}=z\log\left(\frac{z+x}{z}\right).$$

Applying Theorem 1 (ii) gives us the result

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^n+1}{n+1}=\log(1+x).$$

Problem 3. Calculate the sum of the series $\sum_{n=0}^{\infty} a^n \sin nx.$

Set $f(n)=\sin nx$, we have the Z transformation

$$Z\left\{f(n)\right\}=\sum_{n=0}^{\infty}\sin nx.z^{-n}$$

$$=\sum_{n=0}^{\infty}\frac{e^{inx}-e^{-inx}}{2}z^{-n}$$

$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}(e^{ix}z^{-1})^n-\sum_{n=0}^{\infty}(e^{-ix}z^{-1})^n\right]$$

$$=\frac{1}{2}\left(\frac{1}{1-e^{ix}z^{-1}}-\frac{1}{1-e^{-ix}z^{-1}}\right)$$

$$=\frac{z\sin x}{z^2-2z\cos x+1}.$$

According to (3) we get

$$Z\left\{a^n\sin nx\right\}=\frac{az\sin x}{a^2-2az\cos x+z^2}.$$

Therefore, using Theorem 1 (ii) we get

$$\sum_{n=0}^{\infty}a^n\sin nx=\frac{a\sin x}{a^2-2a\cos x+1}.$$

4. Conclusions

Thus, the above result shows that the sum of many series can be calculated through the Z transformation. From there, we see another application of the Z transformation in Mathematics, as well as opening up research suggestions about its other applications.

Declaration of Competing Interest

The author declare no competing interests.

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