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INDECOMPOSABLE MD4-SUBALGEBRAS OF $\mathbf{gl}(3,\mathbb{R})$

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ABSTRACT

In this paper, we introduce all subalgebras of $\mathrm{gl}(3,\mathbb{R})$ *which are 4-dimensional MD-algebras, i.e. the solvable real Lie algebras of dimension 4 such that the co-adjoint orbits of its corresponding connected and simply connected Lie groups are either orbits of dimension zero or orbits of maximal dimension.*

Keywords: Lie algebra, MD-algebra, K-orbits, Matrix algebras.

TÓM TẮT

Các MD4-đại số con bất khả phân của $\mathbf{gl}(3,\mathbb{R})$

Trong bài báo này, chúng tôi giới thiệu tất cả các đại số con của $\mathrm{gl}(3,\mathbb{R})$ *mà đồng thời là các MD-đại số 4 chiều, tức là các đại số Lie thực giải được 4 chiều sao cho quỹ đạo đối phụ hợp của nhóm Lie liên thông, đơn liên tương ứng của chúng có số chiều hoặc là 0 hoặc cực đại.*

Từ khóa: đại số Lie, MD-đại số, K-quỹ đạo, đại số ma trận.

1. Introduction

In the theory of Lie groups and Lie algebras, the Ado's Theorem (see, for example, [3, Section 6.2]) show that any finite dimensional Lie algebra over a field F has always a (linear) faithful representation. In other words, we always can assume that any finite dimensional Lie algebra over F is a subalgebra of the matrix algebra $\mathrm{gl}(n,F)$ of all $(n\times n)$-matrices over F for some $n\in\mathbb{N}\setminus\{0,1\}$. In the theory of representations of Lie algebras, the first problem of interest is to find all the (linear) representations (in particular, faithful representations) of a certain n-dimensional Lie algebra in the matrix algebras. The

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second problem is to find minimal dimensional (linear) faithful representations of a certain *n*-dimensional Lie algebra. Hence, it is very important to find all the subalgebras of the matrix algebras $\mathrm{gl}(n,F)$. Note that, it is easy to list all subalgebras of the algebra $\mathrm{gl}(2,F)$. Recently, G. Thompson and Z. Wick have listed all of subalgebras of $\mathrm{gl}(3,F)$ in 2012 (see [7]).

On the other hand, studying theory of representations, A. A. Kirillov [3] introduced the Orbit Method in 1962. This method quickly became the most important method in the theory of representations of Lie groups and Lie algebras. Using the Kirillov's Orbit Method, we can obtain all the unitary irreducible representations of solvable and simply connected Lie groups and Lie algebras. The importance of Kirillov's Orbit Method is the co-adjoint representation (K-representation). Therefore, it is meaningful to study the K-representation in the theory of representations of Lie groups and Lie algebras. In 1980, Do Ngoc Diep [2] suggested to consider the class of Lie groups and Lie Algebras MD. Let *G* be an *n*-dimensional solvable real Lie group. It is called an MD*n*-group if and only if its orbits in the K-representation (i.e. K-orbits) are orbits of dimension zero or maximal dimension. The corresponding Lie algebra of *G* is called an MD*n*-algebra. There are at least two reasons to consider MD-groups and MD-algebras. The first one, the group C*-algebra of MD-groups can be described by using KK-functors. The second one, for any MD-group *G*, the family of K-orbits of the maximal dimension forms a measured foliation in term of A. Connes. This foliation is called MD-foliation associated to *G*. In addition, the Connes C*-algebra of MD-foliations (see [1]) can be also described by KK-functors (see [2]). Note that all Lie algebras of dimension less than 4 are MD-algebras and they are classified easily. In 1990, Le Anh Vu have classified (up to an isomorphism) all MD-algebras of dimension 4 (see [4] or [2, Chapter 4]). From 2008 to 2012, Le Anh Vu and his colleagues have classified all MD-algebras of dimension 5 in [5], [6]. Recently, Le Anh Vu el al. have classified all MD-algebras of arbitrary dimension such that their derived ideal is 1-dimensional or 1-codimensional in [7].

In this paper, we give some MD-criterias and introduce all 4-dimensional MD-algebras which are subalgebras of $\mathrm{gl}(3,\mathbb{R})$. Furthermore, all 4-dimensional MD-groups which are subgroups of $\mathrm{GL}(3,\mathbb{R})$ are also given.

The paper is organized as follows: The first section introduces the problem studied in the paper. Section 2 deals with some preliminary notions and it is also devoted to the discussion of some MD-criterias. The last section is devoted to the discussion of the main results of the paper.

2. MD-groups and md-algebras

We first recall in this section some preliminary notations and results which will be used later. For details we refer the reader to the book [3] of A. A. Kirillov, the book [2] of Do Ngoc Diep, the papers [4], [5], [6] and [7] of Le Anh Vu el al.

2.1. The K-representation and K-orbits

Let G be a Lie group, $\mathcal{G} = \mathrm{Lie}(G)$ be the corresponding Lie algebra of G and $\mathcal{G}^*$ be the dual space of $\mathcal{G}$. For every $g \in G$, we denote the internal automorphism associated with g by $A_{(g)}$, and whence, $A_{(g)} : G \to G$ can be defined as follows $A_{(g)}(x) := g.x.g^{-1}, \ \forall x \in G$.

This automorphism induces the following map $A_{(g)*} : \mathcal{G} \to \mathcal{G}$ which is defined as follows

$$A_{(g)*}(X) := \frac{d}{dt}\left[g.\exp(tX).g^{-1}\right]\Big|_{t=0}; \ \forall X \in \mathcal{G}..$$

This map is called *the tangent map* of $A_{(g)}$. We now formulate the definitions of K-representation and K-orbits.

Definition 2.1.1. (see [3, Section 6.3]). *The action*

$$Ad : G \to \mathrm{Aut}(\mathcal{G})$$
$$g \mapsto A_{(g)*}$$

is called the adjoint representation of G in $\mathcal{G}$.

Definition 2.1.2. (see [3, Section 6.3]). *The action*

$$K : G \to \mathrm{Aut}(\mathcal{G}^*)$$
$$g \mapsto K_{(g)}$$

such that

$$\left\langle K_{(g)}F, X\right\rangle := \left\langle F, Ad\left(g^{-1}\right)X\right\rangle; \ \left(F \in \mathcal{G}^*, X \in \mathcal{G}\right)$$

is called the co-adjoint representation or K-representation of G in $\mathcal{G}^*$.

Definition 2.1.3. (see [3, Section 6.3]). *Each orbit of the co-adjoint representation of G is called a co-adjoint orbit or K-orbit of G.*

We denote the K-orbit containing F by Ω_F. Thus, for every $F \in \mathcal{G}^*$, the K-orbit containing F defined above can be written by $\Omega_F := \left\{K_{(g)}F \mid g \in G\right\}$. The dimension of every K-orbit of an arbitrary Lie group G is always even. In order to define the dimension of the K-orbits Ω_F for each F from the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G} = \mathrm{Lie}(G)$, it is useful to consider the following skew-symmetric bilinear form B_F on $\mathcal{G}$:

$$B_F(X,Y) = \langle F, [X,Y] \rangle, \ \forall X, Y \in \mathcal{G}.$$

Denote the stabilizer of F under the co-adjoint representation of G in $\mathcal{G}^*$ by G_F and $\mathcal{G}_F := \mathrm{Lie}(G_F)$.

We shall need in the sequel of the following result.

Proposition 2.1.4. (see [3, Section 15.1]). $\mathrm{Ker} B_F = \mathcal{G}_F$; $\dim \Omega_F = \dim \mathcal{G} - \dim \mathcal{G}_F = \mathrm{rank} B_F$. □

2.2. MD-groups, MD-algebras and MD-criteria

In this subsection, we first give the definition of MD-groups, MD-algebras. Next, we give some necessary conditions and one MD-criteria (i.e. necessary and sufficient condition) for MD-algebras.

Definition 2.2.1. (see [2, Chapter 4, Section 4.1]). *An n-dimensional MD-group or, for brevity, an MDn-group is an n-dimensional solvable real Lie group such that its K-orbits are orbits of dimension zero or maximal dimension. The Lie algebra of an MDn-group is called an n-dimensional MD-algebra or, for brevity, an MDn-algebra.*

Proposition 2.2.2. (see [2, Chaper 4, Proposition 1.1]). *Let* $\mathcal{G}$ *be an MD-algebra. Then its second derived ideal* $\mathcal{G}^2 := [[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]]$ *is commutative.* □

We point out here that the converse of the above results is in general not true. In other words, the above necessary condition is not a sufficient condition. So, we now only consider the solvable real Lie algebras having commutative second derived ideals. Thus, they could be MD-algebras.

As an immediate consequence of Proposition 2.1.4, we have the following sufficient and necessary MD-criteria.

Proposition 2.2.3. (see [3, Section 15.1]). *Let* $\mathcal{G}$ *be a solvable real Lie algebra. Then* $\mathcal{G}$ *is an MD-algebra if and only if the rank of matrix*

$$B_F = \begin{pmatrix} F([x_1,x_1]) & F([x_1,x_2]) & \dots & F([x_1,x_n]) \\ F([x_2,x_1]) & F([x_2,x_2]) & \dots & F([x_2,x_n]) \\ \dots & \dots & \dots & \dots \\ F([x_n,x_1]) & F([x_n,x_2]) & \dots & F([x_n,x_n]) \end{pmatrix}$$

is 0 or a constant for every $F \in \mathcal{G}^*$. □

3. Indecomposable MD4-subalgebras of $\mathbf{gl}(3,\mathbb{R})$

In the last section, we give all MD4-subalgebras of the matrix algebra $\mathrm{gl}(3,\mathbb{R})$ of all (3×3)-matrices over $\mathbb{R}$. The corresponding matrix Lie groups in $\mathrm{GL}(3,\mathbb{R})$ are also given. These results are based on the results in [9] of G. Thompson and Z. Wick in 2012.

3.1. List of 4-dimensional Indecomposable Lie Algebras

Firstly, we introduce the list of all 4-dimensional indecomposable Lie algebras which has listed in [7] by J. Patera et al.

Proposition 3.1.1. (see [8]). *Let* $\mathrm{G}=\mathrm{gen}(e_1,e_2,e_3,e_4)$ *be an arbitrary 4-dimensional indecomposable real Lie algebra. Then, up to an isomorphism,* G *is one of the following algebras*

(i) $\mathrm{G}_{4,1}:\ [e_2,e_4]=e_1,\ [e_3,e_4]=e_2;$

(ii) $\mathrm{G}_{4,2a}:\ [e_1,e_4]=ae_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=e_2+e_3,\ (a\neq 0);$

(iii) $\mathrm{G}_{4,3}:\ [e_1,e_4]=e_1,\ [e_3,e_4]=e_2;$

(iv) $\mathrm{G}_{4,4}:\ [e_1,e_4]=e_1,\ [e_2,e_4]=e_1+e_2,\ [e_3,e_4]=e_2+e_3;$

(v) $\mathrm{G}_{4,5ab}:\ [e_1,e_4]=e_1,\ [e_2,e_4]=ae_2,\ [e_3,e_4]=be_3,\ (ab\neq 0,-1\le a\le b\le 1);$

(vi) $\mathrm{G}_{4,6ab}:\ [e_1,e_4]=ae_1,\ [e_2,e_4]=be_2-e_3,\ [e_3,e_4]=e_2+be_3,\ (a\neq 0,b\ge 0);$

(vii) $\mathrm{G}_{4,7}:\ [e_2,e_3]=e_1,\ [e_1,e_4]=2e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=e_2+e_3;$

(viii) $\mathrm{G}_{4,8}:\ [e_2,e_3]=e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=-e_3;$

(ix) $\mathrm{G}_{4,9b}:\ [e_2,e_3]=e_1,\ [e_1,e_4]=(b+1)e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=be_3,\ (-1<b\le 1);$

(x) $\mathrm{G}_{4,10}:\ [e_2,e_3]=e_1,\ [e_2,e_4]=-e_3,\ [e_3,e_4]=e_2;$

(xi) $\mathrm{G}_{4,11a}:\ [e_2,e_3]=e_1,\ [e_1,e_4]=2ae_1,\ [e_2,e_4]=ae_2-e_3,\ [e_3,e_4]=e_2+ae_3,\ (a>0);$

(xii) $\mathrm{G}_{4,12}:\ [e_1,e_3]=e_1,\ [e_2,e_3]=e_2,\ [e_1,e_4]=-e_2,\ [e_2,e_4]=e_1.$ □

Proposition 3.1.2. (see [9, Section 6]). *Up to an isomorphism, there are only three 4-dimensional indecomposable algebras which have a faithful representation in* $\mathrm{gl}(3,\mathbb{R})$. *That are* $\mathrm{G}_{4,8}$, $\mathrm{G}_{4,9b}$ *and* $\mathrm{G}_{4,12}$. □

Remark 3.1.3. (see [4, Theorem 1]). *In view of the listed of all MD4-algebras which has classified in [4], we have the following remark:*

(i) $\mathcal{G}_{4,8}$ *is the real diamond Lie algebra* $\mathrm{Lie}(\mathbb{R}.H_3)$. *This algebra is a semidirect product of the 3–dimensional real Heisenberg Lie algebra* H_3 *by* $\mathbb{R}$.

(ii) $\mathcal{G}_{4,12}$ *is the complex affine Lie algebra* $\mathrm{Lie}(Aff\mathbb{C})$*, i.e. the Lie algebra of the group of all affine transformations of* $\mathbb{C}$.

3.2. Indecomposable MD4-subalgebras of $\mathbf{gl}(3,\mathbb{R})$

The following theorem is the main result of the paper.

Theorem 3.2.1. *Up to an isomorphism, there are exactly two indecomposable MD4-algebras which are subalgebras of* $\mathrm{gl}(3,\mathbb{R})$. *That are the real diamond Lie algebra* $\mathrm{Lie}(\mathbb{R}.H_3)$ *and the complex affine Lie algebra* $\mathrm{Lie}(Aff\mathbb{C})$.

Proof. In view of Proposition 3.1.2 and Remark 3.1.3, $\mathcal{G}_{4,8}=\mathrm{Lie}(\mathbb{R}.H_3)$, $\mathcal{G}_{4,9b}$ and $\mathcal{G}_{4,12}=\mathrm{Lie}(Aff\mathbb{C})$ are all of 4-dimensional indecomposable Lie subalgebras of $\mathrm{gl}(3,\mathbb{R})$. Upon direct computation and applying Proposition 2.2.3 we can see that $\mathcal{G}_{4,9b}$ is not an MD4-algebra. Specifically, the bilinear form B_F on $\mathcal{G}_{4,9b}$ is given as follows:

$$B_F=\begin{pmatrix} 0 & 0 & 0 & (b+1)\alpha \\ 0 & 0 & \alpha & \beta \\ 0 & -\alpha & 0 & b\gamma \\ -(b+1)\alpha & -\beta & -b\gamma & 0 \end{pmatrix}$$

with $F=\alpha e_1^*+\beta e_2^*+\gamma e_3^*+\delta e_4^*;\ \ \alpha,\beta,\gamma,\delta\in\mathbb{R}$.

It is easy to see that

$$\mathrm{rank}(B_F)=\begin{cases} 0, & \alpha=\beta=\gamma=0; \\ 2, & \alpha=\gamma=0,\beta\neq 0; \\ 4, & \beta=\gamma=0,\alpha\neq 0. \end{cases}$$

So, follow Proposition 2.2.3, $\mathcal{G}_{4,9b}$ is not an MD4-algebra.

Therefore only the real diamond Lie algebra $\mathrm{Lie}(\mathbb{R}.H_3)$ and the complex affine Lie algebra $\mathrm{Lie}(Aff\mathbb{C})$ are all MD4-algebras which are subalgebras of $\mathrm{gl}(3,\mathbb{R})$. The proof is complete. □

Corollary 3.2.2. *Up to an isomorphism, there are exactly two indecomposable exponential MD4-group which are subgroups in* $\mathrm{GL}(3,\mathbb{R})$. *They are the real diamond Lie group* $\mathbb{R}.H_3$ *and the complex affine Lie group* $Aff\mathbb{C}$. □

Remark 3.2.3. (see [9]).

(i) In [9], the authors have introduced all n-dimensional subalgebras of $\mathrm{gl}(3,\mathbb{R})$, $0 < n \le 9$.

(ii) Note that any algebras of dimension no more than 3 are MD-algebras. Therefore, any n-dimensional subalgebras of $\mathrm{gl}(3,\mathbb{R})$, $(0 < n \le 3)$ *are MD-subalgebras.*

(iii) In the next paper, we will consider all n-dimensional subalgebras of $\mathrm{gl}(3,\mathbb{R})$, $(5 \le n \le 9)$ *and will prove that all such subalgebras are not MD-algebras.*

We close the paper with the following proposition about the new characteristic of the real diamond Lie algebra $\mathrm{Lie}(\mathbb{R}.H_3)$ and the complex affine Lie algebra $\mathrm{Lie}(Aff\mathbb{C})$.

Proposition 3.2.4. *Let* $\mathcal{G}$ *be an indecomposable MDn-algebra* $(n \ge 4)$. *Then* $\mathcal{G}$ *have a faithful representation in* $\mathrm{gl}(3,\mathbb{R})$ *if and only if* $\mathcal{G}$ *is the real diamond Lie algebra* $\mathrm{Lie}(\mathbb{R}.H_3)$ *or the complex affine Lie algebra* $\mathrm{Lie}(Aff\mathbb{C})$. □

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