ISSN: 2734-9918

TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
Tập 19, Số 6 (2022): 844-855
Website: https://journal.hcmue.edu.vn

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
Vol. 19, No. 6 (2022): 844-855
https://doi.org/10.54607/hcmue.js.19.6.3362(2022)

Research Article

THE BOUNDEDNESS OF CALDERÓN-ZYGMUND OPERATORS OF TYPE THETA ON GENERALIZED WEIGHTED LORENTZ SPACES

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Received: February 08, 2022; Revised: May 08, 2022; Accepted: June 16, 2022

ABSTRACT

In this paper, we consider Calderón-Zygmund operators of type θ (see Definition 1.3 and Definition 1.4 in Section 1) on generalized weighted Lorentz spaces $\Lambda_u^p(w)$, where u is a function that belongs to the class A_p of Muckenhoupt weights on $\mathbb{R}^n$ and w is a function that belongs to the class $B_p(u)$ of Ariño-Muckenhoupt weights on $(0,\infty)$ (see Section 1). In this setting, we first establish the pointwise estimate for the Hardy-Littlewood maximal operator and the sharp maximal operator (see Lemma 2.3 in Section 2) by using Kolmogorov's inequality, Holder's inequality, and the conditions of standard kernels in the definition of Calderón-Zygmund operators of the type θ. Thanks to this significant pointwise estimate, we then prove that Calderón-Zygmund operators of type θ are bounded on the generalized weighted Lorentz spaces $\Lambda_u^p(w)$ (see Theorem 2.4) by employing the ideas and techniques related to maximal operators from the work of Carro et al., (2021). Our main results extend the ones of Carro et al., (2021).

Keywords: Ariño and Muckenhoupt weight; Calderón-Zygmund operator of type θ; eneralized weighted Lorentz space; maximal operator

1. Introduction

It is well-known that the Hardy-Littlewood maximal operator M is bounded on $L_p(u)$ if and only if u belongs to the class A_p of Muckenhoupt weights on $\mathbb{R}^n$ (Muckenhoupt, 1972), where $1 < p < \infty$ and A_p is the class of all weights u satisfying.

Cite this article as: Thai Hoang Minh, Nguyen Van Tien Dat, Hoang Nam Phuong, & Tran Tri Dung (2022). The boundedness of Calderón-Zygmund operators of type theta on generalized weighted Lorentz spaces. *Ho Chi Minh City University of Education Journal of Science, 19*(6), 844-855.

$$\left(\frac{1}{|B|}\int_B u(x)dx\right)\left(\frac{1}{|B|}\int_B \left(u(x)\right)^{\frac{-1}{p-1}}\right)^{p-1} < \infty,$$

for every ball $B \subset \mathbb{R}^n$. Later, this result was extended for the case of the Hilbert transform H and singular integral operators T by Hunt et al. (1973) and Coifman et al. (1974), respectively. More specifically, these authors showed that H and T are bounded on $L^p(u)$ if and only if $u \in A_p$, $1 < p < \infty$.

The weighted Lorentz spaces $\Lambda^p(w)$ were first introduced by Lorentz

(Liu et al., 2002), where w are weights on $(0,\infty)$. The boundedness of M and H on these spaces was investigated by Ariño et al. (1990) and Neugebauer (1992) respectively. In particular, Ariño and Muckenhoupt (1990) indicated that the necessary and sufficient condition of w for the boundedness of M on $\Lambda^p(w)$ is $w \in B_p$, which means

$$r^p \int_r^\infty \frac{w(s)}{s^p} ds \le C \int_0^r w(s)ds, \text{ for every } r > 0.$$

Later, Neugebauer (1992) proved that H is bounded on $\Lambda^p(w)$ if and only if $w \in B_\infty^* \cap B_p$, where B_∞^* is the set of all functions w such that its primitive W holds

$$\int_0^t \frac{W(s)}{s} ds \le CW(t), \text{ for every } t > 0,$$

where $W(t) = \int_0^t w(s)ds$.

In this paper, we consider generalized weighted Lorentz spaces $\Lambda_u^p(w)$, defined as follows (Carro et al., 2007).

Definition 1.1. *Let* u *be a weight on* $\mathbb{R}^n$, w *be a weight on* $(0,\infty)$ *and* $1 < p < \infty$. *The generalized weighted Lorentz space* $\Lambda_u^p(w)$ *is the set of all measurable functions* f *satisfying*

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty f_u^*(t)^p w(t)dt\right)^{1/p} < \infty,$$

where f_u^* *is the decreasing rearrangement of* f, *which is defined as*

$$f_u^*(t) = \inf\left\{s > 0 : \lambda_f^u(s) \le t\right\}, \ t \ge 0,$$

and

$$\lambda_f^u(s) = u\left(\left\{x \in \mathbb{R}^n : |f(x)| > s\right\}\right), \ s > 0$$

is the distribution function of f *with respect to the measure* $u(x)dx$.

Remark 1.2. (Carro et al., 2007, Proposition 2.2.5) The functional $\|f\|_{\Lambda_u^p(w)}$ has the following equivalent expression

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty pt^{p-1}W\left(\lambda_f^u(t)\right)dt\right)^{1/p}.$$

According to Carro et al. (2007), M is bounded on $\Lambda_u^p(w)$ if and only if there exists $q \in (0, p)$ such that for every finite family of cubes $(Q_j)_{j=1}^n$ and for every measurable set $(E_j)_{j=1}^n$, with $E_j \subset Q_j$, for every j, we have

$$\frac{W\left(u\left(\cup_j Q_j\right)\right)}{W\left(u\left(\cup_j E_j\right)\right)} \le C \max_j \left(\frac{|Q_j|}{|E_j|}\right)^q. \tag{1.1}$$

If we denote by $B_p(u)$ the set of all weights w satisfying (1.1) then M is bounded on $\Lambda_u^p(w)$ if and only if $w \in B_p(u)$. Afterward, the characteristic of the boundedness of H was proved by Agora et al. (2013). More recently, Carro et al. (2021) demonstrated that the operator T is bounded on $\Lambda_u^p(w)$ if $u \in A_\infty$ and $w \in B_p(u) \cap B_\infty^*$ where $A_\infty = \bigcup_{p>1} A_p$.

On the other hand, Yabuta (1985) first introduced Calderón-Zygmund operators T of type θ and then showed that T is of strong type (p, p) on $L^p(u)$ for $1 < p < \infty$, and T is of weak type $(1,1)$ on $L^1(u)$. For the convenience of the reader, we recall here the definition of Calderón-Zygmund operators of the type θ.

Definition 1.3. *Let* θ *be a non-negative, non-decreasing function on* $(0,\infty)$, *and* $\int_0^1 \theta(t)t^{-1}dt < \infty$. *A continuous function* $K(x, y)$ *on* $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ *is said to be a standard kernel of type* θ *if it satisfies the following conditions.*

(i) (Size condition)

$$|K(x, y)| \le \frac{C}{|x - y|^n}. \tag{1.2}$$

(ii) (Regularity condition)

$$|K(x, y) - K(x_0, y)| + |K(y, x) - K(y, x_0)| \le C|x_0 - y|^{-n}\, \theta\left(\frac{|x_0 - x|}{|y - x_0|}\right), \tag{1.3}$$

for every x, x_0, y *with* $2|x-x_0|<|y-x_0|$.

Definition 1.4. *Let* θ *be a function in Definition 1.2. A linear operator* T *from* $S(\mathbb{R}^n)$ *to* $S'(\mathbb{R}^n)$ *is said to be a Calderón-Zygmund operator of type* θ *if it satisfies the following conditions.*

(i) T *is bounded on* $L^2(\mathbb{R}^n)$, *that is*

$$\|Tf\|_{L^2} \le C\|f\|_{L^2}, \text{ for every } f \in C_0^\infty(\mathbb{R}^n). \tag{1.4}$$

(ii) There exists a standard kernel K *of type* θ *such that for every function* $f \in C_0^\infty(\mathbb{R}^n)$ *and* $x \notin supp(f)$

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)dy. \tag{1.5}$$

Motivated by the works mentioned above, our aim in this paper is to establish the boundedness of Calderón-Zygmund operators of type θ on $\Lambda_u^p(w)$. Our main result generalizes the one of Carro et al. (2021).

Our paper is organized as follows. In Section 2, we will first present pointwise estimates for Hardy-Littlewood maximal operators and key lemmas. Using these estimates, we then establish the boundedness of Calderón-Zygmund operators of type θ on $\Lambda_u^p(w)$.

As usual, we use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line and C_p to denote a positive constant that is dependent on subscript p. If $f \le Cg$, we write $f \lesssim g$; and if $f \lesssim g \lesssim f$, we write $f \sim g$.

2. Main results

2.1. Pointwise estimates for maximal operator

For $\beta > 0$, let M_β be the modified Hardy–Littlewood maximal function

$$M_\beta f(x) = M(|f|^\beta)^{1/\beta}(x) = \left(\sup_{r>0} \frac{1}{|B|}\int_B |f(y)|^\beta \, dy\right)^{1/\beta},$$

and let $M_\beta^\sharp$ be the modified sharp maximal function

$$M_\beta^\sharp f(x) = \sup_{r>0}\inf_{c\in\mathbb{R}} \left(\frac{1}{|B|}\int_B \left||f(y)|^\beta - |c|^\beta\right| dy\right)^{1/\beta},$$

where $B = B(x,r)$ is a ball in $\mathbb{R}^n$.

Remark 2.1. *It is clear to see that*

$$M^\sharp_\beta(f)(x) \sim \sup_{r>0}\left(\frac{1}{|B|}\int_B \left||f(y)|^\beta - |f|^\beta_B\right| dy\right)^{1/\beta}. \qquad (2.1)$$

Lemma 2.2. (The Kolmogorov's inequality) *(Lu et al., 2007, Theorem 1.3.3) Suppose that T is a sublinear operator from $L^p(\mathbb{R}^n)$ to measurable function spaces and $1 \le p < \infty$.*

(i) If T is of weak type (p,p), then for all $0 < r < p$ and all sets E with finite measure, there exists a constant $C > 0$ such that

$$\frac{1}{|E|}\int_E |Tf(x)|^r \, dx \le C\frac{1}{|E|^{r/p}}\|f\|_p^r. \qquad (2.2)$$

(ii) If there exists $r \in (0,p)$ and a constant $C > 0$ such that (2.2) holds for all sets E with finite measure and $f \in L^p(\mathbb{R}^n)$, then T is of weak type (p,p).

To establish our main result, we need to prove a pointwise estimate for the modified sharp maximal operator. We follow Alvarez et al. (1994) Liu and Lu (2002).

Lemma 2.3. *Let T be a Calderón-Zygmund operator of type θ and $0 < \beta < 1$. Then, there exists a constant $C > 0$ such that*

$$M^\sharp_\beta(Tf)(x_0) \le CMf(x_0),$$

for all bounded functions f with compact support.

Proof. First, we prove for each $0 < \beta < 1$, each ball $B = B(x_0, r)$ and for some constant $c = c_\beta$, there exists $C = C_\beta$ such that

$$\left(\frac{1}{|B|}\int_B \left||Tf|^\beta - |c|^\beta\right| dx\right)^{1/\beta} \le CMf(x_0).$$

Let $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus 2B}$. We pick $c = (Tf_2)_B$, it follows from the following inequalities

$$\left||a|^s - |b|^s\right| \le |a-b|^s,$$

$$|a+b|^s \lesssim |a|^s + |b|^s,$$

that

$$\left(\frac{1}{|B|}\int_B \left||Tf|^\beta - |(Tf_2)_B|^\beta\right| dx\right)^{1/\beta} \le \left(\frac{1}{|B|}\int_B \left|Tf - (Tf_2)_B\right|^\beta dx\right)^{1/\beta}$$

$$= \left(\frac{1}{|B|}\int_B \left|Tf_1 + \left(Tf_2 - (Tf_2)_B\right)\right|^\beta dx\right)^{1/\beta}$$

$$\lesssim\left(\frac{1}{|B|}\int_B\left(\left|Tf_1\right|^\beta+\left|Tf_2-\left(Tf_2\right)_B\right|^\beta\right)dx\right)^{1/\beta}$$

$$\lesssim\left(\frac{1}{|B|}\int_B|Tf_1|^\beta\,dx\right)^{1/\beta}$$

$$+\left(\frac{1}{|B|}\int_B|Tf_2-\left(Tf_2\right)_B|^\beta\,dx\right)^{1/\beta}.$$

Set $I_1=\left(\frac{1}{|B|}\int_B|Tf_1|^\beta\,dx\right)^{1/\beta}$ and $I_2=\left(\frac{1}{|B|}\int_B|Tf_2-\left(Tf_2\right)_B|^\beta\,dx\right)^{1/\beta}$. Since T is an operator of weak type $(1,1)$ and $0<\beta<1$, so according to the Kolmogorov's inequality, we get

$$\frac{1}{|B|}\int_B|Tf_1|^\beta\,dx\lesssim\left(\frac{1}{|B|}\int_{\mathbb{R}^n}|f_1(x)|\,dx\right)^\beta.$$

This implies

$$I_1\lesssim\frac{1}{|B|}\int_{\mathbb{R}^n}|f_1(x)|\,dx=\frac{1}{|B|}\int_{2B}|f|\,dx\lesssim\frac{1}{|2B|}\int_{2B}|f|\,dx\le Mf(x_0).$$

For I_2, applying Holder's inequality with $0<\beta<1$ yields

$$I_2=\frac{1}{|B|^{1/\beta}}\frac{1}{|B|^{1-1/\beta}}\left(\int_B|Tf_2-(Tf_2)_B|^\beta\,dx\right)^{1/\beta}\left(\int_B dx\right)^{1-1/\beta}$$

$$\leqslant\frac{1}{|B|}\int_B|Tf_2-(Tf_2)_B|\,dx$$

$$\leqslant\frac{1}{|B|}\int_B\left|\int_{\mathbb{R}^n}K(x,y)f_2(y)dy-\frac{1}{|B|}\int_B\int_{\mathbb{R}^n}K(z,y)f_2(y)dydz\right|dx$$

$$\leqslant\frac{1}{|B|}\int_B\left|\frac{1}{|B|}\int_B\int_{\mathbb{R}^n}K(x,y)f_2(y)dydz-\frac{1}{|B|}\int_B\int_{\mathbb{R}^n}K(z,y)f_2(y)dydz\right|dx$$

$$\leqslant\frac{1}{|B|}\frac{1}{|B|}\int_B\left|\int_B\int_{\mathbb{R}^n}f_2(y)\left(K(x,y)-K(z,y)\right)dydz\right|dx$$

$$\leqslant\frac{1}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\mathbb{R}^n}|f_2(y)|\left|K(x,y)-K(z,y)\right|dydzdx.$$

Take $z,\ x\in B$ and $y\notin 2B$. Then, we have $2|x-x_0|<|y-x_0|$ and $2|z-x_0|<|y-x_0|$. It follows from the regularity condition of the definition of the standard kernel of type θ that

$$\int_{\mathbb{R}^n}|f_2(y)|\left|K(x,y)-K(x_0,y)\right|dy=\int_{\mathbb{R}^n\setminus 2B}|f(y)|\left|K(x,y)-K(x_0,y)\right|dy$$

$$\lesssim \sum_{j=1}^{\infty} \int_{2^{j+1}B\backslash 2^j B} \frac{\theta(|x-x_0|/|y-x_0|)}{|x_0-y|^n} |f(y)|dy$$

$$\lesssim \sum_{j=1}^{\infty} \theta(2^{-j}) \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|dy$$

$$\lesssim \sum_{j=1}^{\infty} \theta(2^{-j}) 2^j Mf(x_0) \lesssim \int_0^1 \frac{\theta(t)}{t} dt Mf(x_0) \lesssim Mf(x_0).$$

By an argument analogous to $|K(z,y)-K(x_0,y)|$, we obtain

$$\int_{\mathbb{R}^n} |f_2(y)| \left|K(z,y)-K(x_0,y)\right| \lesssim Mf(x_0)dy.$$

On the other hand, it is clear to see that

$$|K(x,y)-K(z,y)| \leqslant |K(x,y)-K(x_0,y)| + |K(z,y)-K(x_0,y)|.$$

This leads to

$$I_2 \lesssim \frac{1}{|B|}\frac{1}{|B|} \int_B \int_B \int_{\mathbb{R}^n} |f_2(y)| \left|K(x,y)-K(z,y)\right| dydzdx$$

$$\lesssim \frac{1}{|B|}\frac{1}{|B|} \int_B \int_B \int_{\mathbb{R}^n} |f_2(y)| (|K(x,y)-K(x_0,y)| + |K(z,y)-K(x_0,y)|) dydzdx$$

$$\lesssim \frac{1}{|B|}\frac{1}{|B|} \int_B \int_B Mf(x_0) dzdx \lesssim Mf(x_0).$$

Hence,

$$\left(\frac{1}{|B|}\int_B \left| |Tf|^\beta - |c|^\beta \, dx \right| \right)^{1/\beta} \lesssim I_1 + I_2 \lesssim Mf(x_0),$$

which completes the proof of Lemma 2.3.

2.2. Main results

In this section, we will prove the following main result.

Theorem 2.4. *Let T be a Calderón-Zygmund operator of type θ, $1 < p < \infty$, $u \in A_\infty$ and $w \in B_\infty^* \cap B_p(u)$. Then T is bounded on $\Lambda_u^p(w)$.*

To prove the main result, we need the following lemmas.

Lemma 2.5. *(Carro et al., 2021, Lemma 2.6) Assume that $1 < p < \infty$, $u \in A_\infty$, $w \in B_\infty^*$, and W satisfies the doubling condition. Then*

$$\| Mf \|_{\Lambda_u^p(w)} \lesssim \| M^\sharp f \|_{\Lambda_u^p(w)},$$

provided that $\| Mf \|_{\Lambda_u^p(w)} < \infty$.

Lemma 2.6. *(Carro et al., 2021, Lemma 2.7) If* $w \in B_\infty^*$ *then there exists* $\varepsilon > 0$ *such that*

$$\int_0^1 \frac{W(t)}{t^{1+\varepsilon}} < \infty.$$

Lemma 2.7. *(Carro et al., 2007, Proposition 2.2.12 and Lemma 3.3.1) If* $w \in B_p(u)$ *then W holds the following conditions:*

(i) $W(2t) \lesssim W(t), \forall t > 0$,

(ii) $W(s+t) \lesssim W(s) + W(t), \forall s, t > 0$.

Now, let us prove the main result by following the ideas by Carro et al. (2021).

Proof of Theorem 2.4. Let f be a bounded function with compact support, $u \in A_\infty$, $w \in B_\infty^* \cap B^p(u)$ and $0 < \beta < 1$. Note that $|f(x)| \leqslant |M_\beta(x)|$ a.e. on $\mathbb{R}^n$. Therefore

$$\| Tf \|_{\Lambda_u^p(w)} \leqslant \| M_\beta(Tf) \|_{\Lambda_u^p(w)}.$$

Since $w \in B_p(u)$, by Lemma 2.7, we see that its primitive W satisfies the doubling condition.

Therefore, in view of Lemma 2.5, if $\| M_\beta T(f) \| < \infty$ then

$$\| M_\beta(Tf) \|_{\Lambda_u^p(w)} \lesssim \| M_\beta^\sharp(Tf) \|_{\Lambda_u^p(w)}.$$

On the other hand, by Lemma 2.3, we obtain

$$\| M_\beta^\sharp(Tf) \|_{\Lambda_u^p(w)} \lesssim \| Mf \|_{\Lambda_u^p(w)}.$$

Since $w \in B_p(u)$, so M is bounded on $\Lambda_u^p(w)$, that is

$$\| Mf \|_{\Lambda_u^p(w)} \lesssim \| f \|_{\Lambda_u^p(w)}.$$

By the above arguments, if $\| M_\beta(Tf) \| < \infty$ then

$$\| Tf \|_{\Lambda_u^p(w)} \leqslant \| M_\beta(Tf) \|_{\Lambda_u^p(w)} \lesssim \| M_\beta^\sharp(Tf) \|_{\Lambda_u^p(w)}$$
$$\lesssim \| Mf \|_{\Lambda_u^p(w)} \lesssim \| f \|_{\Lambda_u^p(w)}.$$

Moreover, since $w \in B_p(u)$, so $w \in B_{p/\beta}(u)$. As a result, M is bounded on $\Lambda_u^{p/\beta}(w)$ and

$$\| M_\beta(Tf) \|_{\Lambda_u^p(w)} = \| M(|Tf|^\beta) \|_{\Lambda_u^{p/\beta}(w)}^{1/\beta} \lesssim \| |Tf|^\beta \|_{\Lambda_u^{p/\beta}(w)}^{1/\beta} = \| Tf \|_{\Lambda_u^p(w)}.$$

At this stage, to complete the proof, we only need to show that $\| Tf \|_{\Lambda_u^p(w)} < \infty$.

Since f is a bounded function with compact support, there exists $R > 0$ such that $B = B(0, R) \subset \operatorname{supp}(f)$. By Lemma 2.7, we have that

$$
\begin{aligned}
& W(u(\{x\in\mathbb{R}^n : |Tf(x)|>t\})) \\
\lesssim\ & W(u(\{x\in 3B : |Tf(x)|>t\})) + W(u(\{x\in\mathbb{R}^n\setminus 3B : |Tf(x)|>t\})).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
& \|Tf\|_{\Lambda_u^p(w)} = \left(\int_0^\infty pW(u(\{x\in\mathbb{R}^n : |Tf(x)|>t\}))t^{p-1}\right)^{1/p} \\
& \lesssim \left(\int_0^\infty W(u(\{x\in 3B : |Tf(x)|>t\}))t^{p-1}\right)^{1/p} \\
& + \left(\int_0^\infty W(u(\{x\in\mathbb{R}^n\setminus 3B : |Tf(x)|>t\}))t^{p-1}\right)^{1/p} \\
& = I_1 + I_2.
\end{aligned}
$$

Now, we prove that I_2 is finite. Assume $x\notin 3B$, $y\in B$. Then we have that $|x|\geqslant 3R$, $|x-y|\geqslant |x|-R\geqslant |x|/2$ and $B\subset B(0,|x|)$. These estimates, together with the definition of standard kernel of type θ of T, imply

$$
\begin{aligned}
|Tf(x)| &= \int_{\mathbb{R}^n} K(x,y)f(y)dy \lesssim \int_B |x-y|^{-n}|f(y)|dy \\
&\lesssim \frac{2^n}{|x|^n}\int_{B(0,|x|)}|f(y)|dy \lesssim \frac{2^{2n}}{(2|x|)^n}\int_{B(0,2|x|)}|f(y)|dy \lesssim Mf(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_2 &= \left(\int_0^\infty W(u(\{x\in\mathbb{R}^n\setminus 3B : |Tf(x)|>t\}))t^{p-1}\right)^{1/p} \\
&\lesssim \left(\int_0^\infty W(u(\{x\in\mathbb{R}^n\setminus 3B : |Mf(x)|>t\}))t^{p-1}\right)^{1/p} \lesssim \|Mf\|_{\Lambda_u^p(w)}.
\end{aligned}
$$

Since $w\in B_p(u)$, M is bounded on $\Lambda_u^p(w)$. Hence, we obtain

$$
\|Mf\|_{\Lambda^p(w)} \lesssim \|f\|_{\Lambda_u^p(w)}.
$$

Then, we deduce that $I_2 \lesssim \|f\|_{\Lambda_u^p(w)} < \infty$.

Next, we prove I_1 is finite. Firstly, we have the following estimates

$$
\begin{aligned}
I_1^p &= \int_0^\infty W\left(\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx\right)t^{p-1}dt \\
&\lesssim \int_0^1 W\left(\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx\right)t^{p-1}dt + \int_1^\infty W\left(\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx\right)t^{p-1}dt \\
&\lesssim \int_0^1 W\left(\int_{3B} u(x)dx\right)t^{p-1}dt + \int_1^\infty W\left(\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx\right)t^{p-1}dt \\
&\lesssim W\left(u(3B)\right) + \int_1^\infty W\left(\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx\right)t^{p-1}dt.
\end{aligned}
$$

It is easy to see that $W(u(3B)) < \infty$. Since $u \in A_\infty$, there exists r' such that $u \in A_{r'}$. Then we can pick $r > \max\{1, r'\}$ such that

$$\int_0^1 \frac{W(t)}{t^{p/r+1}} dt < \infty.$$

On the other hand, it is well-known that if $u \in A_r$ then T is of week type (r,r) on $L^r(u)$. Therefore, for every $t > 0$,

$$\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx \lesssim \int_{\{x\in\mathbb{R}^n:|Tf(x)|>t\}} u(x)dx \lesssim \frac{\| f \|^r_{L^r(u)}}{t^r},$$

which leads to

$$\int_1^\infty W\left(\int_{\{x\in 3B:|Tf(x)|>t\}} u(x)dx\right) t^{p-1}dt \lesssim \int_1^\infty W\left(\frac{\| f \|^r_{L^r(u)}}{t^r}\right) t^{p-1}dt$$

$$\lesssim \int_1^\infty W\left(\frac{1}{t^r}\right) t^{p-1}dt \approx \int_0^1 \frac{W(t)}{t^{p/r+1}} dt < \infty,$$

where we use Lemma 2.7 in the first two inequalities.

3. Conclusion

In summary, using the ideas and techniques of Carro et al. (2021), we obtain the following pointwise estimate for the Hardy-Littlewood maximal operator, the sharp maximal operator (Lemma 2.3), and the boundedness of Calderón-Zygmund operators of type θ on the generalized weighted Lorentz spaces $\Lambda_u^p(w)$ (Theorem 2.4):

Main Result 1. *Let T be a Calderón-Zygmund operator of type θ and $0 < \beta < 1$. Then, there exists a constant $C > 0$ such that*

$$M_\beta^\sharp(Tf)(x_0) \le CMf(x_0),$$

for all bounded functions f with compact support.

Main Result 2. *Let T be a Calderón-Zygmund operator of type θ, $1 < p < \infty$, $u \in A_\infty$ and $w \in B_\infty^* \cap B_p(u)$. Then T is bounded on $\Lambda_u^p(w)$.*

These two results generalize what Carro et al. (2021) demonstrated. We will investigate the boundedness of commutators of Calderón-Zygmund operators of type θ on the spaces $\Lambda_u^p(w)$ in the forthcoming paper.

- ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

- ***Acknowledgements:*** *This research is funded by Ho Chi Minh City University of Education under the CS.2021.19.03TĐ project.*

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TÍNH BỊ CHẶN CỦA TOÁN TỬ CALDERÓN-ZYGMUND LOẠI THETA TRÊN KHÔNG GIAN LORENTZ TỔNG QUÁT

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Ngày nhận bài: 08-02-2022; ngày nhận bài sửa: 08-5-2022; ngày duyệt đăng: 16-6-2022

TÓM TẮT

Trong bài báo này, chúng tôi xét các toán tử Calderón-Zygmund loại θ *(xem Định nghĩa 1.3 và Định nghĩa 1.4 trong Phần 1) trên không gian Lorentz có trọng tổng quát* $\Lambda_u^p(w)$, *trong đó* u *là một hàm thuộc lớp hàm trọng Muckenhoupt trên* $\mathbb{R}^n$ *và* w *là một hàm thuộc lớp hàm trọng Ariño-Muckenhoupt* $B_p(u)$ *trên* $(0,\infty)$ *(xem Phần 1). Trong cấu hình này, chúng tôi thiết lập đánh giá từng điểm cho toán tử cực đại Hardy-Littlewood và toán tử cực đại nhọn (xem Bổ đề 2.3 trong Phần 2) bằng cách sử dụng bất đẳng thức Kolmogorov, bất đẳng thức Holder và các điều kiện của nhân chuẩn trong định nghĩa các toán tử Calderón-Zygmund loại* θ. *Nhờ vào đánh giá từng điểm quan trọng này, từ đó chúng tôi chứng minh rằng các toán tử Calderón-Zygmund loại* θ *bị chặn trên không gian Lorentz có trọng tổng quát* $\Lambda_u^p(w)$ *(xem Định lí 2.4) bằng cách vận dụng các ý tưởng và kĩ thuật liên quan đến toán tử cực đại trong công trình của Carro và cộng sự (2021). Các kết quả chính nêu trên của chúng tôi mở rộng các kết quả tương ứng trong bài báo của Carro và cộng sự (2021).*

Từ khóa: hàm trọng Ariño Muckenhoupt; toán tử Calderón-Zygmund loại θ; không gian Lorentz có trọng tổng quát; toán tử cực đại