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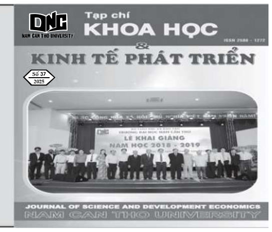

On the stability of set optimization problems via upper set less order relations

Lam Van Day[1*], Nguyen Mai Nhat Duong[2] , Nguyen Tien Doanh[3]

[1]*Department of Mathematics, Faculty of Basic Sciences, Nam Can Tho University, Can Tho City, Vietnam*

[2]*Che Guevara High School, Mo Cay Commune, Vinh Long Province*

[3]*Ngo Si Lien Secondary school, Vinh Thong Ward, An Giang Province*

[*]*Corresponding author: Lam Van Day (email: lvday@nctu.edu.vn)*

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ABSTRACT

This paper aims to establish stability conditions, in the Painlevé–Kuratowski sense, for solution sets of set optimization problems under perturbations in both the constraint set and the objective mapping. By relaxing the assumptions of continuity and compactness of the objective mapping, we investigated external stability for weakly efficient and efficient solutions. Moreover, we employed the dominance property of set optimization problems under these conditions to analyze internal stability for efficient solutions. Our results represent a new contribution to the field.

1. INTRODUCTION

Stability conditions of solutions for set optimization problems play an important role in the stability theory of optimization models. In practice, the original problem may be subject to perturbations caused by various factors, leading to the consideration of sequences of problems with data approximating the original data. In such cases, stability conditions aim to ensure that, from a certain point onward, approximate solutions remain sufficiently close to the exact solution of the original problem. In fact, since model data are often obtained through approximation methods such as statistical analysis or measurement, researchers tend to focus more on the stability of approximate solutions, as these are of greater practical relevance than exact solutions.

The above issues highlight the need to study optimization models whose objective functions take set-valued outputs. For this class of problems, two main approaches are currently recognized. Among them, the approach based on order relations between sets is considered effective and has attracted significant attention from researchers in recent years. This approach was initiated by the work of Kuroiwa (1998). Subsequently, Jahn and Ha (2011) combined two strategies namely, the vector approach and the set-order approach to propose new order relations that enhance the rigor of comparing sets. These are fairly complex order relations, and to date, no studies have analyzed the solution structure or qualitative properties of solutions to problems using this type of ordering. Due to technical limitations and the complexity of computational tools, results regarding convergence conditions for set optimization models remain relatively limited.

In this paper, we focus on studying and developing results and approaches related to stability conditions for set optimization problems, based on research published in recent years. Specifically, Gutiérrez et al. (2016) used conditions related to the continuity and domination property of the set optimization problems, as well as the compactness of the constraint set, to analyze the external stability of the weakly efficient solution set in the sense of Hausdorff convergence, as well as the internal stability of the efficient solution set in the sense of Hausdorff and Painlevé–Kuratowski convergence in the image space, for set optimization problems with perturbed constraint sets. Karuna and Lalitha (2019) applied conditions concerning the closedness, convexity, and sequential compactness of the constraint set, as well as Hausdorff continuity, compact-valuedness, strict quasiconvexity via cone, and domination property of the set optimization problems, to investigate the internal stability of the efficient solution set in the sense of Hausdorff and Painlevé–Kuratowski convergence, and the weakly efficient solution set in the Painlevé–Kuratowski sense. In addition, they employed these tools to study the internal stability, in the sense of Hausdorff and Painlevé–Kuratowski convergence, for the efficient solution set, and in the Painlevé–Kuratowski sense for the weakly efficient solution set in the image space of set optimization problems with perturbed constraint sets. Most recently, Han (2022) examined both external and internal stability of the efficient and weakly efficient solution sets by applying conditions related to cone convexity, compactness, and strict connectedness via cone of the objective function in the decision space, in the case of problems perturbed in both the objective function and the constraint set.

We observe that, with the techniques and tools proposed above, by suitably adjusting the set order relations and the approach, we can relax certain assumptions while still obtaining results comparable to those in Han (2022) Moreover, when considering special cases, our results prove to be stronger than existing ones, which will be discussed in detail in the main results section.

2. RESEARCH METHODS

In this paper, we applied analytical, synthetic, and interpretative methods to conduct the research. Based on a review of relevant literature and the achievements in this field in recent years, we have established several new results.

3. RESULTS AND DISCUSSION

Throughout this paper, unless otherwise stated, we assume that $\mathbb{X}, \mathbb{Y}$ are normed vector spaces. We denote by $B_{\mathbb{Y}}$ the closed unit ball in $\mathbb{Y}$. Let C be a convex, closed, pointed cone with vertex at the origin in $\mathbb{Y}$. Let $\mathcal{P}(\mathbb{Y})$ denote the family of all nonempty subsets of $\mathbb{Y}$. For any $A \in \mathcal{P}(\mathbb{Y})$, we denote by $\mathrm{int}A, \mathrm{cl}A$, and $\mathrm{bd}A$ the interior, closure, and boundary of the set A, respectively.

Given $M, N \in \mathcal{P}(\mathbb{Y})$, we consider the upper set less order relations $\preceq_u$ and $\prec_u$, defined as follows

$$A \preceq_u B \Leftrightarrow A \subseteq B - C,$$

and

$$A \prec_u B \Leftrightarrow A \subseteq B - \text{int}C.$$

The equivalence relation $\sim_u$ on $\mathcal{P}(\mathbb{Y})$ is defined by

$$A \sim_u B \Leftrightarrow A \preceq_u B \text{ and } B \preceq_u A.$$

For convenience in the following sections, we denote $\preceq_u, \prec_u$, and $\sim_u$ simply by $\preceq$, $\prec$, and $\sim$, respectively.

Definition 1. (Avriel *et al.,* 2010) Let A be a nonempty subset of a topological space. The set A is called arcwise-connected if for any two points $x, y \in A$, there exists a continuous function $\gamma: [0,1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Let F be a set-valued mapping from $\mathbb{X}$ to $\mathbb{Y}$. The domain of F is defined by

$$\text{dom}F = \{x \in \mathbb{X}: F(x) \neq \emptyset\}.$$

Let that $X \subset \text{dom}F$ is a nonempty subset of X. We consider the following set optimization problem:

$$\text{(SOP)} \ \min F(x) \text{ với } x \in X.$$

Definition 2. An element $x_0 \in X$ is called: (i) *an efficient solution* of (SOP) if

$$x \in X, F(x) \preceq F(x_0) \Rightarrow F(x_0) \preceq F(x).$$

(ii) *a weakly efficient solution* of (SOP) if

$$x \in X, F(x) \prec F(x_0) \Rightarrow F(x_0) \prec F(x).$$

We denote by $\text{Eff}(F, X)$ and $\text{WEff}(F, X)$ the sets of efficient solutions and weakly efficient solutions of (SOP), respectively. Let $x \in X$. We define the equivalence set associated with x as follows

$$[x]_F = \{y \in X: F(y) \sim F(x)\}$$

Proposition 1. If $x_0 \in \text{Eff}(F, X)$, then $[x_0]_F \subset \text{Eff}(F, X)$.

Proof. Take an arbitrary element $y \in [x_0]_F$. We will prove that y is an efficient solution of problem (SOP). Since $y \in [x_0]_F$, it follows that $F(y) \preceq F(x_0)$. Let z be any element in X such that $F(z) \preceq F(y)$. Combining this with $F(y) \preceq F(x_0)$ and using the transitivity of $\preceq$, we obtain $F(z) \preceq F(x_0)$. Since $x_0 \in \text{Eff}(F, X)$ and $F(z) \preceq F(x_0)$, we have $F(x_0) \preceq F(z)$. Combining $F(x_0) \preceq F(z)$ with $F(y) \preceq F(x_0)$ once again, we get $F(y) \preceq F(z)$. Therefore, y is an efficient solution of problem (SOP).

Definition 3. (Han, 2022) Let X be an arcwise-connected subset of $\mathbb{X}$. A set-valued mapping F from $\mathbb{X}$ to $\mathbb{Y}$ is said to be C-strictly quasiconnected on X if, for every $A \in \mathcal{P}(\mathbb{Y})$ and for all $u, v \in X$ with $u \neq v, F(u) \preceq A$ and $F(v) \preceq A$, there exists a continuous path $\gamma: [0,1] \to X$ with $\gamma(0) = u$ and $\gamma(1) = v$ such that

$$F(\gamma(t)) \prec A, \forall t \in (0,1).$$

Definition 4. (Han and Huang, 2018) Let X be a nonempty and convex subset of $\mathbb{X}$. A set-valued mapping F from $\mathbb{X}$ to $\mathbb{Y}$ is said to be strictly naturally quasi C-convex if, for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ and for all $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subset F(t x_1 + (1 - t) x_2) + \text{int}C$$

Remark 1**.** (Han, 2022)

It is clear that if $F: \mathbb{X} \rightrightarrows \mathbb{Y}$ is strictly naturally quasi C-convex on X, then F is also C-strictly quasiconnected on X.

Lemma 1. (Han, 2022)

Assume that X is arcwise connected and F is C-strictly quasiconnected on X with nonempty compact values. Then,

$$\text{WEff}(F, X) = \text{Eff}(F, X)$$

Next, we recall the concept of Painlevé-Kuratowski convergence for a sequence of subsets.

Definition 5. (Khan *et al.,* 2016, page 83) Let $\{A_n\}$ be a sequence of nonempty subsets of X. The lower and upper limits in the sense of PainlevéKuratowski are defined respectively as follows:

$$\mathrm{Li}A_n := \left\{x \in X : x = \lim_{n\to\infty} x_n, x_n \in A_n, \text{ for } n \text{ sufficiently large } \right\},$$

$$\mathrm{Ls}A_n := \left\{x \in X : x = \lim_{s\to\infty} x_s, x_s \in A_{n_s}, \{n_s\} \text{ is a subsequence of } \{n\}\right\}.$$

If $\mathrm{Ls}(A_n) \subseteq A$, then the sequence $\{A_n\}$ is said to *Painlevé-Kuratowski upper converge* to A, denoted $A_n \overset{K}{\rightharpoonup} A$. If $A \subseteq \mathrm{Li}(A_n)$, the sequence $\{A_n\}$ is said to *Painlevé-Kuratowski lower converge* to A, denoted $A_n \overset{K}{\rightarrow} A$. If $\mathrm{Ls}A_n \subset A \subset \mathrm{Li}A_n$ then $\{A_n\}$ is said to *Painlevé-Kuratowski converge* to A, denoted $A_n \overset{K}{\rightarrow} A$.

Now, we recall a commonly used concept of convergence, namely Hausdorff convergence. A sequence $\{A_n\}$ of nonempty subsets of $\mathbb{X}$ is said to *Hausdorff converge* to a nonempty subset A of $\mathbb{X}$ (denoted $A_n \overset{\mathrm{H}}{\rightarrow} A$) if

$$e(A_n, A) \to 0 \text{ and } e(A, A_n) \to 0,$$

where

$$e(A,B) := \sup_{a\in A} d(a,B) = \inf\{\lambda \geq 0 : A \subseteq B + \lambda B_{\mathbb{X}}\},$$

and

$$d(a,B) := \inf_{b\in B} \|b-a\|_{\mathbb{X}}$$

with B a nonempty subset of $\mathbb{X}$. We say that $\{A_n\}$ Hausdorff upper converge to A when $e(A_n, A) \to 0$ (denoted $A_n \overset{H}{\rightharpoonup} A$), and Hausdorff lower converge when $e(A, A_n) \to 0$ (denoted $A_n \overset{H}{\rightarrow} A$).

Lemma 2. (Peng *et al.,* 2023) assume that A is a compact set and $A_n \overset{H}{\rightharpoonup} A$. For any sequence $x_n \in A_n, \forall n$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in A$.

Definition 6. (Khan *et al.,* 2016, page 83) Let $\mathbb{X}$ and $\mathbb{Y}$ be two topological vector spaces. A set-valued mapping F from $\mathbb{X}$ to $\mathbb{Y}$ is said to be

(a) *C-upper semicontinuous* (C-usc) at $x_0 \in \mathbb{X}$ if, for any neighborhood V of $F(x_0)$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \subset V - C, \forall x \in U(x_0).$$

(b) *C-lower semicontinuous* (C-lsc) at $x_0 \in \mathbb{X}$ if, for any $y \in F(x_0)$ and any neighborhood V of $0_{\mathbb{Y}}$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \cap (y + V + C) \neq \emptyset, \forall x \in U(x_0).$$

(c) *Hausdorff C-upper semicontinuous* (H-C-usc) at $x_0 \in \mathbb{X}$if, for any neighborhood$V$ of $\theta_{\mathbb{Y}}$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \subset F(x_0) + V - C, \forall x \in U(x_0).$$

(d) *Hausdorff C-lower semicontinuous* (H-C-lsc) at $x_0 \in \mathbb{X}$ if, for any neighborhood$V$ of $\theta_{\mathbb{Y}}$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x_0) \subset F(x) + V - C, \forall x \in U(x_0).$$

Mapping F is said to be H − C-usc, H − C-lsc, C -usc, and C -lsc on $\mathbb{X}$ if it satisfies these respective properties at every point $x \in \mathbb{X}$. F is said to be C -continuous on $\mathbb{X}$ if it is both C -usc and C -lsc on $\mathbb{X}$.

Next, we consider the following important result.

Lemma 3. Assume that $F(x_0) - C$ is closed and $F(x_0) - C$ is proper. Then, $x_0 \in WEff(F, X)$ if and only if there does not exist $x \in X$ such that $F(x) \prec F(x_0)$.

Proof: Suppose there exists $x \in X$ such that $F(x) \prec F(x_0)$. By the definition of relation $\prec$, we have

$$F(x) \subset F(x_0) - \mathrm{int}C.$$

Since x_0 is assumed to be a weakly efficient solution, according to its definition, we have $F(x_0) \prec F(x)$, tức là $F(x_0) \subseteq F(x) - intC$. This together with $F(x) \subseteq F(x_0) - intC$ gives us

$F(x_0) \subseteq F(x) - intC \subseteq F(x_0) - intC - intC \subseteq F(x_0) - intC$. (1)

Combining (1) with $F(x_0) - intC \subseteq F(x_0) - C$, we obtain $F(x_0) - C = F(x_0) - intC$. This leads to a contradiction. The proof is completed.

Proposition 2. Let $F: \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping. The following statements hold:

(a) If F is C-usc at x_0, then F is H - C-usc at x_0.

(b) If $F(x_0)$ is compact and F is H - C-usc at x_0, then F is C-usc at x_0.

(c) If F is H - C-lsc at x_0, then F is C-lsc at x_0.

(d) If $F(x_0)$ is compact and F is C-lsc at x_0, then F is $H - C$-lsc at x_0.

Proof: Statements (a), (b), and (c) can be easily verified. Now, we prove statement (d). Assume, on contrary, that F is not H - C-lsc at x_0. Then there exists a neighborhood V_0 of $0_{\mathbb{Y}}$ such that, for any neighborhood U of x_0, there exists $\hat{x} \in U$ satisfying:

$$F(x_0) \nsubseteq F(\hat{x}) + V_0 - C.$$

This means that there exists a sequence $\{x_n\} \subset \mathbb{X}$ with $x_n \to x_0$ such that

$$F(x_0) \nsubseteq F(x_n) + V_0 - C,$$

and thus, there exists $y_n \in F(x_0)$ such that

$y_n \notin F(x_n) + V_0 - C$. (2)

Since $F(x_0)$ is compact, without loss of generality, assume $y_n \to y_0 \in F(x_0)$. For the given V_0, there exists a balanced neighborhood V_1 of $0_{\mathbb{Y}}$ such that

$V_1 + V_1 \subset V_0$. (3)

Because F is C-lsc at x_0, applying the definition at $y_0 \in F(x_0)$ and for V_1, there exists a neighborhood U_0 of x_0 such that

$$F(x) \cap (y_0 + V_1 + C) \neq \emptyset, \forall x \in U_0.$$

Since $x_n \to x_0$, we have $x_n \in U_0$ for large enough n. Hence

$$F(x_n) \cap (y_0 + V_1 + C) \neq \emptyset.$$

Thus, there exist $z_n \in F(x_n), v_n \in V_1$, and $c_n \in C$ such that

$$z_n = y_0 + v_n + c_n.$$

Since V_1 is balanced, it follows that

$y_0 = z_n - v_n - c_n \in F(x_n) + V_1 - C$. (4)

Because $y_n \to y_0$, we get $y_n \in y_0 + V_1$ for n sufficiently large. This along with (3 và (4), we yield

$y_n \in y_0 + V_1 \subset F(x_n) + V_1 - C + V_1 \subset F(x_n) + V_0 - C$, which contradits to (2).

Definition 7. (Peng *et al.,* 2023) Let F_n be set -valued mapping from $\mathbb{X}$ into $\mathbb{Y}$. Let A be a nonempty subset of $\mathbb{X}$ with $A \subseteq \mathrm{dom}(F_n) \cap \mathrm{dom}(F)$ for all $n \in \mathbb{N}$. We say that

(a) $F_n \overset{\mathrm{HCl}}{\to} F$ on A if, for any neighborhood V of $0_{\mathbb{Y}}$, there exists $n_0 \in \mathbb{N}$ such that

$$F(x) \subseteq F_n(x) + V - C, \forall x \in A, \forall n \geq n_0.$$

(b) $F_n \overset{\mathrm{HCu}}{\to} F$ on A if, for any neighborhood V of $0_{\mathbb{Y}}$, there exists $n_0 \in \mathbb{N}$ such that

$$F_n(x) \subseteq F(x) + V - C, \forall x \in A, \forall n \geq n_0.$$

We say that $F_n \overset{\mathrm{HC}}{\to} F$ (also called H-C convergence) on A, if both $F_n \overset{\mathrm{HCl}}{\to} F$ and $F_n \overset{\mathrm{HCu}}{\to} F$ hold on A.

For convenience in studying the main results, we now consider the following lemma. Lemma 3. Let $\{x_n\} \subset A$ satisfy $x_n \to x_0$. Then the following statements hold:

(i) If F is H-C-usc at x_0 and $F_n \overset{\mathrm{HCu}}{\to} F$ on A, then for any neighborhood V of $0_{\mathbb{Y}}$, there exists $n_0 \in \mathbb{N}$ such that

$$F_n(x_n) \subseteq F(x_0) + V - C, \forall n \geq n_0.$$

(ii) If F is H-C-lsc at x_0 and $F_n \overset{\text{HCl}}{\to} F$ on A, then for any neighborhood V of $0_{\mathbb{Y}}$, there exists $n_0 \in \mathbb{N}$ such that

$$F(x_0) \subseteq F_n(x_n) + V - C, \forall n \geq n_0.$$

Proof. By the similar argument, we only need to show the case (ii). For any neighborhood V of $0_{\mathbb{Y}}$, there exists a neighborhood W of $0_{\mathbb{Y}}$ such that:

$$W + W \subset V. \quad (5)$$

Since F is H-C-lsc at x_0, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x_0) \subseteq F(x) + W - C, \forall x \in U(x_0).$$

Since $x_n \to x_0$, there exists $n_1 \in \mathbb{N}$ such that $x_n \in U(x_0)$ for all $n \geq n_1$. This together with H-C-lower semicontinuity of F gives us

$$F(x_0) \subseteq F(x_n) + W - C, \forall n \geq n_1. \, (6)$$

Because $F_n \overset{\text{HCl}}{\to}$ F on A, there exists $n_2 \in \mathbb{N}$ such that

$$F(x) \subseteq F_n(x) + W - C, \forall x \in A, \forall n \geq n_2. \quad (7)$$

Set $n_0 = \max\{n_1, n_2\}$. From $x_n \in A$ and (7), we have

$$F(x_n) \subseteq F_n(x_n) + W - C, \forall n \geq n_0. \ (8)$$

For all $n \geq n_0$, from (5), (6), and (8), it follows that

$$\begin{aligned} F(x_0) &\subseteq F(x_n) + W - C \\ &\subseteq F_n(x_n) + W - C + W - C \\ &\subseteq F_n(x_n) + V - C. \end{aligned}$$

This completes the proof.

External stability of the set optimization problem:

Before analyzing the external stability of the efficient and weakly efficient solution sets, we present the following important results. Lemma 4. Let $\{(x_n, y_n)\} \subset A \times A, (x_n, y_n) \to (x_0, y_0)$. Assume that

(i) $F(y_0)$ is compact and $F(y_0) \prec F(x_0)$;

(ii) F is H-C-usc at y_0 and H-C-lsc at x_0;

(iii) $F_n \overset{HC}{\longrightarrow} F$ on A.

Then, there is $n_0 \in \mathbb{N}$ satisfying

$$F_n(y_n) \prec F_n(x_n), \qquad \forall n \geq n_0.$$

Proof. Suppose, by contradiction, that for every $k \in \mathbb{N}$, there exists $n_k \geq k$ such that

$$F_{n_k}(y_{n_k}) \nprec F_{n_k}(x_{n_k}).$$

Without loss of generality, we can write

$$F_n(y_n) \nprec F_n(x_n), \forall n,$$

which means that for all n

$$F_n(y_n) \nsubseteq F_n(x_n) - \text{int}C.$$

Therefore, there exists $v_n \in F_n(y_n)$ such that

$$v_n - u_n \notin -\text{int}C, \forall u_n \in F_n(x_n). \ (9)$$

Since F is $\text{H} - C$-usc at y_0 and $y_n \to y_0$, for every $\varepsilon > 0$, we have

$$v_n \in F_n(y_n) \subseteq F(y_0) + \varepsilon B_{\mathbb{Y}} - C, \text{ for all sufficiently large } n.$$

This means that for large n, there exist $\hat{v}_n \in F(y_0)$ and $c_n^1 \in C$ such that

$$v_n - \hat{v}_n + c_n^1 \in \varepsilon B_{\mathbb{Y}}. \quad (10)$$

On the other hand, since $F(y_0)$ is compact, there exists a subsequence of $\{\hat{v}_n\}$ (still denoted $\{\hat{v}_n\}$) such that $\hat{v}_n \to \hat{v}_0 \in F(y_0)$. Thus, for large n

$$\hat{v}_n - \hat{v}_0 \in \varepsilon B_{\mathbb{Y}}. \ (11)$$

From (10) and (11), it follows that

$$v_n + c_n^1 - \hat{v}_0 = (v_n + c_n^1 - \hat{v}_n) + (\hat{v}_n - \hat{v}_0) \in 2\varepsilon B_{\mathbb{Y}}, \text{ for large } n. \quad (12)$$

Since $\hat{v}_0 \in F(y_0)$ and $F(y_0) \prec F(x_0)$, there exists $u_0 \in F(x_0)$ such that

$$\hat{v}_0 - u_0 \in -\text{int}C. \qquad (13)$$

From $u_0 \in F(x_0)$, $x_n \to x_0$ and F is H-C-lsc at x_0, it follows that

$$u_0 \in F(x_0) \subseteq F_n(x_n) + \varepsilon B_{\mathbb{Y}} - C, \text{ for large } n.$$

Therefore, for large n, there exist $u_n \in F_n(x_n)$ and $c_n^2 \in C$ such that

$$u_0 - u_n + c_n^2 \in \varepsilon B_{\mathbb{Y}}. \quad (14)$$

Combining this with (12) and (13), for sufficiently large n we obtain:

$$\begin{aligned} v_n + c_n^1 - (u_n - c_n^2) &= [v_n + c_n^1 - \hat{v}_0] + [\hat{v}_0 - u_0] \\ &\quad + [u_0 - (u_n - c_n^2)] \\ &\in 3\varepsilon B_{\mathbb{Y}} - \text{int}C. \end{aligned}$$

Now, we need to prove that there exists n_0 such that

$$v_n + c_n^1 - (u_n - c_n^2) \in -\text{int}C, \forall n \geq n_0$$

Otherwise, for every k there exists $n_k \geq k$ such that

$$v_{n_k} + c_{n_k}^1 - \left(u_{n_k} - c_{n_k}^2\right) \in \mathbb{Y} \setminus (-\text{int}C)$$

Letting $k \to +\infty$, by (12), (14), and the closedness of $\mathbb{Y} \setminus (-\text{int}C)$, we obtain:

$$\hat{v}_0 - u_0 \in \mathbb{Y} \setminus (-\text{int}C)$$

which contradicts (13). Hence, we conclude

$$\exists n_0 \in \mathbb{N}: v_n + c_n^1 - (u_n - c_n^2) \in -\text{int}C, \forall n \geq n_0.$$

This is equivalent to

$$\begin{aligned} v_n - u_n &\in -\text{int}C - c_n^1 - c_n^2 \\ &\subseteq -\text{int}C, \text{ for large } n, \end{aligned}$$

which contradicts to (9). Therefore, the lemma is proved.

Lemma 5. Let $\{(x_n, y_n)\} \subset A \times A, (x_n, y_n) \to (x_0, y_0)$. Assume that

(i) $F(x_0) - C$ is closed and $F_n(y_n) \preceq F_n(x_n)$ for all $n \in \mathbb{N}$;

(ii) F is $\text{H} - C$-lsc at y_0 and $\text{H} - C$-usc at x_0;

(iii) $F_n \overset{HC}{\to} F$ on A.

Then, $F(y_0) \preceq F(x_0)$.

Proof. From the assumption, we have $F_n(y_n) \subseteq F_n(x_n) - C, \forall n \in \mathbb{N}$. (15)

Since $F_n \overset{\text{HCl}}{\to} F$ on A and F is $\text{H} - C$-lsc at y_0, by Lemma 3 (ii), for any $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $F(y_0) \subseteq F_n(y_n) + \varepsilon B_Y - C, \forall n \geq n_1$. (16)

Similarly, since $F_n \overset{\text{HCu}}{\to} F$ on A and F is H-C-usc at x_0, by Lemma 3(i), for any $\varepsilon > 0$, there exists $n_2 \in \mathbb{N}$ such that $F_n(x_n) \subseteq F(x_0) + \varepsilon B_Y - C, \forall n \geq n_2$. (17)

Combining (15), (16), and (17), for all $n \geq n_0 := \max\{n_1, n_2\}$, we obtain

$$\begin{aligned} F(y_0) &\subseteq F_n(y_n) + \varepsilon B_Y - C \\ &\subseteq F_n(x_n) - C + \varepsilon B_Y - C \\ &\subseteq F(x_0) + 2\varepsilon B_Y - C. \end{aligned}$$

Since ε is arbitrary and $F(x_0) - C$ is closed, it follows that

$$F(y_0) \subseteq F(x_0) - C.$$

Thus, the proof is completed.

Now, let X_n and X be nonempty subsets of $\mathbb{X}$, with $\bigcup_{n \in \mathbb{N}} X_n \subset A$ and $X \subset A$. Then, we consider the perturbed set optimization problem

$$(\text{SOP})^n: \min_{x \in X_n} F_n(x).$$

Similarly, we define the corresponding solution sets.

$$\text{Eff}(F_n, X_n) := \{x_0 \in X_n \mid \forall x \in X_n, [F_n(x) \preceq F_n(x_0)] \Rightarrow [F_n(x_0) \preceq F_n(x)]\},$$
$$\text{WEff}(F_n, X_n) := \{x_0 \in X_n \mid \forall x \in X_n, [F_n(x) \prec F_n(x_0)] \Rightarrow [F_n(x_0) \prec F_n(x)]\}.$$

Below, we present one of our main results in this paper.

Theorem 1. Assume that

(i) F is $\text{H} - C$-continuous and has compact values on X;

(ii) $X_n \overset{K}{\rightharpoonup} X$ and $F_n \overset{HC}{\to} F$ on A.

Then, $\text{LsWEff}(F_n, X_n) \subseteq \text{WEff}(F, X)$.

Moreover, if X is arcwise-connected and F is C-strictly quasiconnected on X, then $\text{LsEff}(F_n, X_n) \subset \text{Eff}(F, X)$.

Proof. First, we prove that $\text{LsWEff}(F_n, X_n) \subseteq \text{WEff}(F, X)$.

Indeed, assume that $x_0 \in \text{LsWEff}(F_n, X_n)$. Then, there exists a subsequence $\{x_n\}$ (still denoted by $\{x_n\}$) with $x_n \in \text{WEff}(F_n, X_n)$ and $x_n \to x_0$. Since $X_n \xrightarrow{K} X$, we have $x_0 \in X$. Assume, by contradiction, that $x_0 \notin \text{WEff}(F, X)$. Then, by Lemma 2, there exists $y_0 \in X$ such that $F(y_0) \prec F(x_0)$,

which implies $F(y_0) \subseteq F(x_0) - \text{int}C$.

Since $y_0 \in X$ and $X \subseteq \liminf X_n$, there exists a sequence $\{y_n\} \subset X_n$ such that $y_n \to y_0$. From (18) and Lemma 4, we deduce that there exists $n_0 \in \mathbb{N}$ such that $F_n(y_n) \subseteq F_n(x_n) - \text{int}C, \forall n \geq n_0$.

By (19) and Lemma 2, it follows that: $x_n \notin \text{WEff}(F_n, X_n)$,

which contradicts the assumption $x_n \in \text{WEff}(F_n, X_n)$. Thus $\text{LsWEff}(F_n, X_n) \subseteq \text{WEff}(F, X)$.

Next, we prove that

$$\text{LsEff}(F_n, X_n) \subseteq \text{Eff}(F, X).$$

Indeed, from Lemma 1, we have $\text{WEff}(F, X) = \text{Eff}(F, X)$

Noting that$\text{Eff}(F_n, X_n) \subseteq \text{WEff}(F_n, X_n)$,

it follows that

$$\text{LsEff}(F_n, X_n) \subseteq \text{LsWEff}(F_n, X_n)$$

Therefore:

$$\text{LsEff}(F_n, X_n) \subseteq \text{LsWEff}(F_n, X_n) \subseteq \text{WEff}(F, X) = \text{Eff}(F, X)$$

This completes the proof.

Remark 2. To the best of our knowledge, the result in Theorem 1 is entirely new, as no prior studies have considered the solution stability of set optimization problems based on the $\preceq^u$ relation. Therefore, the subsequent results derived from this theorem are also new. Our findings contribute to enriching the research landscape on solution stability for set-valued optimization problems.

Example 1. Let $\mathbb{X} = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, X = [0,1], X_n = [0, 1 + 1/n]$, and consider the set-valued maps $F, F_n: \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined for all n by

$$F(x) = \begin{cases} \{x\} \times [0,1], & \text{if } x < 1, \\ \{(1,1)\}, & \text{if } x = 1, \\ \{x\} \times [0, x], & \text{if } x > 1, \end{cases}$$

and

$$F_n(x) = \begin{cases} \{x\} \times [0, 1 + 1/n], & \text{if } x < 1, \\ \{(1, 1 + 1/n)\}, & \text{if } x = 1, \\ \{x\} \times [0, x + 1/n], & \text{if } x > 1. \end{cases}$$

It is clear that the assumptions of Theorem 1 are satisfied, so the theorem can be applied. By direct computation, we obtain

$\text{WEff}(F_n, X_n) = \{0\} = \text{Eff}(F_n, X_n)$, and $\text{WEff}(F, E) = \{0\} = \text{Eff}(F, E)$. Hence, Theorem 1 holds.

Corollary 1. Assume that

(i) F is H-C-continuous and compact-valued on X;

(ii) $X_n \xrightarrow{K} X$ and $F_n \xrightarrow{HC} F$ on A.

Then, $\text{LsWEff}(F_n, X_n) \subseteq \text{WEff}(F, X)$.

Moreover, if X is convex and F is strictly naturally quasi C-convex on X, then

$$\text{LsEff}(F_n, X_n) \subseteq \text{Eff}(F, X).$$

Corollary 2. Suppose that $F_n \equiv F$, and the following assumptions are satisfied:

(i) F is H-C-continuous and compact-valued on X;

(ii) $X_n \xrightarrow{K} X$.

Then,

$$\text{LsWEff}(F, X_n) \subseteq \text{WEff}(F, X).$$

Moreover, if X is arcwise-connected and F is C-strictly quasiconnected on X, then

$$\text{LsEff}(F, X_n) \subseteq \text{Eff}(F, X).$$

Corollary 3. Suppose that $F_n \equiv F$ and the following assumptions are satisfied

(i) F is H-C-continuous and compact-valued on X;

(ii) $X_n \overset{K}{\rightharpoonup} X$.

Then,

$$\text{LsWEff}(F, X_n) \subseteq \text{WEff}(F, X).$$

Moreover, if X is convex and F is strictly naturally quasi C-convex on X, then

$$\text{LsEff}(F, X_n) \subseteq \text{Eff}(F, X).$$

Corollary 4. Suppose that $X_n = X$ for all n and the following assumptions hold

(i) F is H-C-continuous and compact-valued on X;

(ii) $F_n \overset{HC}{\rightarrow} F$ on A.

Then,

$$\text{LsWEff}(F_n, X) \subseteq \text{WEff}(F, X).$$

Moreover, if X is arcwise-connected and F is C-strictly quasiconnected on X, then

$$\text{LsEff}(F_n, X) \subseteq \text{Eff}(F, X).$$

Corollary 5. Suppose that $X_n = X$ for all n, and the following assumptions hold

(i) F is H-C-continuous and compact-valued on X;

(ii) $F_n \overset{\text{HC}}{\rightarrow} F$ on A.

Then,

$$\text{LsWEff}(F_n, X) \subseteq \text{WEff}(F, X).$$

Moreover, if X is convex and F strictly naturally quasi C-convex on X, then

$$\text{LsEff}(F_n, X) \subseteq \text{Eff}(F, X).$$

Internal stability of the set optimization problem:

To study the internal stability of problem (SOP), we first recall the definition of the domination property of (SOP).

Definition 8. (Anh *et al.,* 2024) For each $n \in \mathbb{N}$, the problem $(SOP)^n$ is said to have the domination property if, for every $x \in X_n$, either $x \in \text{Eff}(F_n, X_n)$ or there exists $z \in \text{Eff}(F_n, X_n)$ such that

$$F_n(z) \preceq F_n(x).$$

Theorem 2. Assume that

(i) F is H-C-continuous and C-closed valued on X;

(ii) $X_n \overset{H}{\rightharpoonup} X, X_n \overset{K}{\rightharpoonup} X$, and X is compact;

(iii) $F_n \overset{HC}{\rightarrow} F$ on A;

(iv) the problem $(SOP)^n$ has the domination property.

Then, for every $x_0 \in \text{Eff}(F, X)$, one of the following two cases holds:

(a) There exists a sequence $\{x_n\}$ with $x_n \in \text{Eff}(F_n, X_n)$ such that $x_n \to x_0$.

(b) There exists a sequence $\{z_n\}$ with $z_n \in \text{Eff}(F_n, X_n)$ having a subsequence $\{z_{n_k}\}$ converging to $z_0 \in [x_0]_F$.

Proof: Since $x_0 \in X$ and $X \subseteq \text{Li}X_n$, there exists a sequence $\{x_n\}$ with $x_n \in X_n$ such that $x_n \to x_0$. Due to the dominance property of problem $(SOP)^n$, one of the following cases occurs

(a) If $x_n \in \text{Eff}(F_n, X_n)$ for all n, the proof is complete.

(b) Otherwise, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \notin \text{Eff}(F_{n_k}, X_{n_k})$. In this case, there exists a sequence $\{z_{n_k}\}$ with $z_{n_k} \in \text{Eff}(F_{n_k}, X_{n_k})$ for all $k \in \mathbb{N}$, such that:

$$F_{n_k}(z_{n_k}) \preceq F_{n_k}(x_{n_k}), \forall k \in \mathbb{N}.$$

Since $X_n \overset{K}{\rightharpoonup} X$ and X is compact, by Lemma 1, there exists a subsequence of $\{z_{n_k}\}$ (without loss of generality, still denoted $\{z_{n_k}\}$) converging to some $z_0 \in X$. Applying Lemma 5, we obtain

$$F(z_0) \preceq F(x_0).$$

This means that $z_0 \in [x_0]_F$. The proof is completed.

Corollary 6**.** Suppose that $F_n \equiv F$ and

(i) F is H-C-continuous and C -closed valued on X;

(ii) $X_n \overset{H}{\to} X, X_n \overset{K}{\to} X_{\mathrm{f}}$ and X is compact;

(iii) the problem $(SOP)^n$ has the domination property.

Then, for every $x_0 \in \mathrm{Eff}(F, X)$, one of the following two cases holds:

(a) There exists a sequence $\{x_n\}$ with $x_n \in \mathrm{Eff}(F, X_n)$ such that $x_n \to x_0$.

(b) There exists a sequence $\{z_n\}$ with $z_n \in \mathrm{Eff}(F, X_n)$, admitting a subsequence $\{z_{n_k}\}$ converging to $z_0 \in [x_0]_F$.

Corollary 7**.** Suppose that $X_n = X$ and:

(i) F is H − C-continuous and C -closed valued on X;

(ii) X is compact;

(iii) $F_n \overset{HC}{\to} F$ on A_i

(iv) the problem $(SOP)^n$ has the domination property.

Then, for every $x_0 \in \mathrm{Eff}(F, X)$, one of the following two cases holds:

(a) There exists a sequence $\{x_n\}$ with $x_n \in \mathrm{Eff}(F_n, X)$ such that $x_n \to x_0$.

(b) There exists a sequence $\{z_n\}$ with $z_n \in \mathrm{Eff}(F_n, X_n)$, admitting a subsequence $\{z_{n_k}\}$ converging to $z_0 \in [x_0]_F$.

4. CONCLUSION

In this paper, we employ the notion of Painlevé–Kuratowski convergence to investigate the stability of solution sets for set-valued optimization problems under simultaneous perturbations of both the objective mapping and the constraint set. By relaxing assumptions related to cone-based Hausdorff semicontinuity, compactness, closedness, strict C-convexity, and strict C-connectedness, we obtain results concerning the external stability of efficient and weakly efficient solution sets. Moreover, we introduce the dominance property and combine it with the aforementioned relaxed conditions to analyze the internal stability of the efficient solution set of the considered set-valued optimization problem.

REFERENCES

Anh, L. Q., Day, L. V., Duy, T. Q., Tammer, C., & Yao, J.C. (2024). External and internal stability for set optimization problems. *Submitted for publication*.

Avriel, M., Diewert, W. E., Schaible, S., & Zang, I. (2010). *Generalized concavity*. Society for Industrial and Applied Mathematics.

Gutiérrez, C., Miglierina, E., Molho, E., Novo, & V. (2016). Convergence of solutions of a set optimization problem in the image space. *Journal of Optimization Theory and Applications, 170*, 358-371.

Han, Y. (2022). Painlevé–Kuratowski convergences of the solution sets for set optimization problems with cone-quasiconnectedness. *Optimization, 71*(7), 2185-2208.

Han, Y., Huang, N. J. (2018). Continuity and convexity of a nonlinear scalarizing function in set optimization problems with applications. *Journal of Optimization Theory and Applications, 177*, 679-695.

Jahn, J., Ha, T. X. D. (2011). New order relations in set optimization. *Journal of*

Optimization Theory and Applications, 148(2), 209-236.

Karuna, Lalitha, C. S. (2019). External and internal stability in set optimization. *Optimization, 68*(4), 833-852.

Khan, A. A., Tammer, C., & Zălinescu, C. (2016). *Set-Valued Optimization*. Springer-Verlag Berlin An.

Kuroiwa, D. (1998). The natural criteria in set-valued optimization. *RIMS Kokyuroku Kyoto Univ* (1031), 85–90.

Peng, Z. Y., Chen, X. J., Zhao, Y. B., & Li, X. B. (2023). Painlevé-Kuratowski convergence of minimal solutions for set-valued optimization problems via improvement sets. *Journal of Global Optimization, 87*(2), 759-781.