EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A STRONGLY DAMPED WAVE EQUATION WITH POLYNOMIAL NONLINEARITIES

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Abstract: This paper studies the first initial boundary value problem for the strongly damped wave equations on a bounded domain $\Omega \subset \mathbb{R}^3$. Under the polynomial growth rate of the nonlinearity *f* and an external force *g* independent on time *t*, we prove the existence and uniqueness of a weak solution. Finally, we show the existence of a weak global attractor for the continuous semigroup generated by the weak solution.

Keywords: Strongly damped wave equations; polynomial growth rate; weak solution; global attractor.

SỰ TỒN TẠI VÀ DÁNG ĐIỆU TIỆM CẬN NGHIỆM CỦA PHƯƠNG TRÌNH TRUYỀN SÓNG TẮT DẦN MẠNH

Tóm tắt: Trong bài báo này chúng tôi xét bài toán biên ban đầu cho lớp phương trình truyền sóng tắt dần mạnh trên miền bị chặn $\Omega \subset \mathbb{R}^3$. Dưới giả thiết hàm phi tuyến *f* tăng trưởng kiểu đa thức và ngoại lực *g* không phụ thuộc thời gian *t*, chúng tôi chứng minh sự tồn tại và duy nhất nghiệm yếu. Cuối cùng, chúng tôi chứng minh sự tồn tại của tập hút toàn cục yếu cho nửa nhóm liên tục kết nối với nghiệm yếu bài toán.

Từ khoá: Phương trình truyền sóng tắt dần mạnh; tăng trưởng kiểu đa thức; nghiệm yếu; tập hút toàn cục.

1. INTRODUCTION

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system having some dissipativity properties is to analyse the existence and structure of its global attractor. Once the global attractor is obtained, a next natural question is to study the most important properties of the global attractor, such as dimension, dependence on parameters, regularity of the attractor,... In the last decades, many authors have paid attention to these problems and obtained many results for a large class of PDEs (see e.g. [3, 9, 10] and references therein).

In this paper we study the following semilinear strongly damped wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} u_{tt} - \omega\Delta u_t - \Delta u + f(u) = g, & x \in \Omega, t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u_t(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1.1}$$

where Δ is a Laplacian with respect to the variable $x \in \Omega$, $u = u(t,x)$ is an unknown function, $\omega > 0$ is a fixed positive number, $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$, $u_1 \in L^2(\Omega)$, $g \in L^2(\Omega)$ are given external forces, $f \in C^1(\mathbb{R})$ are known functions satisfying the following conditions:

$$C_1 |u|^p - C_0 \quad \leq f(u)u \leq C_2 |u|^p + C_0, p \geq 2, \tag{1.2}$$

$$C_3 |u|^{p-2} - C_3 \quad \leq f'(u) \leq C_3 |u|^{p-2} + C_3, p \geq 2, \tag{1.3}$$

for some C_0, C_1, C_2, C_3 are all positive.

In this paper, we conduct a comprehensive study of Eq. (1.1) within a 3D bounded domain with a smooth boundary, addressing the well-posedness of weak energy solutions and the existence of global attractors.

2. LITERATURE REVIEW, THEORETICAL FRAMEWORK AND METHODOLOGY

The semilinear strongly damped wave equations are quite interesting from a physical viewpoint. For example, they arise in the modeling of the flow of viscoelastic fluids as well as in the theory of heat conduction.

In the cases of one and two spatial dimensions, (1.1) models the transverse vibrations of a homogeneous string and the longitudinal vibrations of a homogeneous bar, respectively, under the influence of viscous effects. The term $-\Delta u_t$ indicates that the stress is proportional not only to the strain, as prescribed by Hooke's law, but also to the strain rate, as observed in a linearized Kelvin-Voigt material. In the three-dimensional context, (1.1) describes the deviation from the rest configuration of a homogeneous and isotropic linearly viscoelastic solid with short memory, categorized as a rate-type material, under the influence of an external displacement-dependent force $g - f(u)$. If the solid is additionally subjected to dynamic friction, the term βu_t with $\beta > 0$ appears on the left-hand side of the equation. It is important to note that even in the case of $n = 1$, the problem (1.1) may fail to admit a global classical solution if the viscosity term $\omega \Delta u_t$ is omitted. Therefore, the inclusion of this damping term serves as a regularization mechanism, preventing the blow-up of solutions.

As is well-known, one of the earliest significant results concerning the global behavior of solutions to problem (1.1) was obtained by Webb [11]. He demonstrated that if $\Omega \subset \mathbb{R}^3$ and the nonlinear term satisfies standard dissipativity conditions (without any growth restrictions), then problem (1.1) admits a unique strong solution in the space $(H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$. Furthermore, each strong solution converges to the corresponding stationary solution as $t \to \infty$. Consequently, it has been established for quite some time that Eq. (1.1) is well-posed and dissipative within the class of strong solutions (specifically, in the phase space $[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$), without requiring any growth restrictions on the nonlinearity f (only the natural quasi-monotonicity condition $f' \geq -K$ is needed; see, e.g. [11]).

However, the scenario appears fundamentally different for less regular energy solutions. The assumption (1.2), together with this condition, indicates that the natural phase space for energy solutions of (1.1) is as follows:

$$\varepsilon = \varepsilon(p) = [H_0^1(\Omega) \cap L^p(\Omega)] \times L^2(\Omega), \qquad \eta_u(t) = (u(t), \partial_t u(t)) \in \varepsilon .$$

It should be noted that in the “subcritical” case $p \leq 6$, the energy phase space $\varepsilon(p)$ is independent of p (due to the embedding $H_0^1(\Omega) \to L^6(\Omega)$) and coincides with the energy space for the linear problem

$$\varepsilon(p) = H_0^1(\Omega) \times L^2(\Omega), \qquad p \le 6,$$

whereas in the "supercritical" case $p > 6$, the energy phase space $\varepsilon(p)$ critically depends on the growth exponent p.

The existence of such solutions can be readily established, for example, by employing the Galerkin method, without imposing any restriction on the exponent p. Nevertheless, the standard approach guarantees uniqueness only when $p \le 6$, and ensures asymptotic smoothness as well as the existence of a global attractor only for $p < 6$ (see [4, 5, 6] and references therein). The case $p = 6$ has been considered more intricate and has only recently been fully understood (see [2, 8, 11]). Thus, to the best of our knowledge, the growth exponent $p = 6$ has generally been regarded as critical for establishing the uniqueness and asymptotic regularity of energy solutions for (1.1). In particular, the well-posedness of energy solutions for problem (1.1) with model nonlinearities like (1.2) has remained an open question.

The paper is organized as follows. In Section 3, we prove the existence and uniqueness of a weak solution using the compactness method. The existence of a weak global attractor $\mathcal{A}$ for the semigroup $S(t)$ generated by (1.1) is demonstrated in Section 4.

Notations. The $L^2(\Omega)$ -norm and the $H_0^1(\Omega)$ -norm will be denoted by $|\cdot|$ and $\|\cdot\|$, respectively, and the norm on $L^p(\Omega)$ by $\|\cdot\|_{L^p}$ and the notation $H^s = D(-\Delta^{s/2})$ for the scale of Hilbert spaces generated by the Laplacian with the Dirichlet boundary. The constant C is different from each appearance. Let X_1, X_2 be two Banach spaces and Z be a topological vector space such that $X_1 \to Z, X_2 \to Z$. Then $X_1 \cap X_2$ and $X_1 + X_2$ are two Banach spaces equipped with the norms

$$\|u\|_{X_1 \cap X_2} = \|u\|_{X_1} + \|u\|_{X_2},$$
$$\|u\|_{X_1 + X_2} = \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2} : u = u_1 + u_2\}.$$

It is known that if $X_1 \cap X_2$ is dense both in X_1 and X_2 then $(X_1 \cap X_2)^* = X_1^* + X_2^*$.

3. THE EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION

Definition 3.1. A weak energy solution of the problem (1.1) on $(0,T)$ is a function $u = u(t,x)$ iff

$$u \in L^{\infty}(0,T;H_0^1(\Omega) \cap L^p(\Omega)),$$
$$u_t \in L^{\infty}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$
$$u_{tt} \in L^2(0,T;H^{-1}(\Omega)) + L^{p'}(0,T;L^{p'}(\Omega)),$$
$$u|_{t=0} = u_0 \quad u_t|_{t=0} = u_1 \text{ a.e in } \Omega,$$

and

$$\int_{\Omega_T} \left(u_{tt}\varphi + \omega \nabla u_t \nabla \varphi + \nabla u \nabla \varphi + f(u)\varphi - g\varphi\right) dxdt = 0,$$

for all test $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$.

We note that, as usual, the trajectory $t \to \eta_u(t) := (u(t), \partial_t u(t))$ is a weaky continuous with respect to t as the $\varepsilon(p)$ -valued function. By this reason, the intial data at $t = 0$ is well defined.

Theorem 3.1. *Under conditions (1.2) and (1.3), problem (1.1) has a unique weak solution* $(u(t), u_t(t))$ *satisfying the dissipative estimate*

$$|u_t|^2 + |\nabla u|^2 + \|u\|_{L^p(\Omega)}^p + \mu \int_0^T |\nabla u_t|^2 \, dt \le C(\|(u_0,u_1)\|_{\varepsilon(p)}, |g|).$$

Proof. (i) Existence. We look for an approximate solution $(u^n(t), u_t^n(t))$ that belongs to the finite-dimensional space spanned by the first n eigenfunctions of the operator $-\Delta$ such that

$$u^n(t) = \sum_{j=1}^{n} u^{n_j}(t) e_j(x),$$

and solves the following problem

$$\langle u_{tt}^n, e_j \rangle + \omega \langle \nabla u_t^n, \nabla e_j \rangle + \langle \nabla u^n, \nabla e_j \rangle + \langle f(u^n), e_j \rangle = (g, e_j), 1 \le j \le n,$$
$$(u^n(0), e_j) = (u_0, e_j),$$
$$(u_t^n(0), e_j) = (u_1, e_j).$$

By the Peano theorem, we obtain the existence of approximate solutions $u^n(t)$. We now establish some *a priori* estimates for (u^n, u_t^n). Multiplying Eq. (1.1) by αu^n, where α is a small positive number which will be fixed below, and integrating over $x \in \Omega$, we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}\alpha\omega|\nabla u^n|^2+\alpha\langle u_t^n,u^n\rangle\Big)-\alpha|u_t^n|^2+\alpha|\nabla u^n|^2+\alpha\langle f(u^n),u^n\rangle=\alpha(g,u^n).$$

(3.1)

Multiplying Eq. (1.1) by u_t^n and integrating over $x\in\Omega$, we arrive at

$$\frac{d}{dt}\Big(\frac{1}{2}|u_t^n|^2+\frac{1}{2}|\nabla u^n|^2+\big(F(u^n),1\big)-(g,u^n)\Big)+\omega|\nabla u_t^n|^2=0.$$

(3.2)

Here $F(u):=\int_0^u f(s)ds$ and $(\cdot,\cdot)$ stands for the usual inner product in $L^2(\Omega)$.

Take a sum of (3.1) and (3.2), denote

$$E\big(u^n(t)\big)=\frac{1}{2}|u_t^n|^2+(\frac{1}{2}\alpha\omega+\frac{1}{2})|\nabla u^n|^2+\big(F(u^n),1\big)+\alpha\langle u_t^n,u^n\rangle-(g,u^n).$$

We deduce that

$$\frac{d}{dt}E(u^n(t))-\alpha|u_t^n|^2+\alpha|\nabla u^n|^2+\alpha\langle f(u^n),u^n\rangle+\omega|\nabla u_t^n|^2=\alpha(g,u^n).$$

Use (1.2), Cauchy's and Poincare's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}(1-\frac{\alpha}{\lambda_1\varepsilon})|u_t^n|^2+\frac{1}{2}\alpha(\omega-\varepsilon)|\nabla u^n|^2+C_1'\|u^n\|_{L^p(\Omega)}^p-C_0'|\Omega|-\frac{1}{2\lambda_1}|g|^2\\&\le E(u^n(t))\\&\le\frac{1}{2}(1+\frac{\alpha}{\lambda_1\varepsilon})|u_t^n|^2+(\frac{\alpha\omega}{2}+\frac{\alpha\varepsilon}{2}+1)|\nabla u^n|^2+C_2'\|u^n\|_{L^p(\Omega)}^p+C_0'|\Omega|+\frac{1}{2\lambda_1}|g|^2,\end{aligned}$$

where λ_1 is the first eigenvalue of Δ in Ω with the homogeneous Dirichlet condition (note that $\|u\|^2\ge\lambda_1|u|^2$).

Fixing $\alpha<\lambda_1\varepsilon$ and $\varepsilon<\omega$ we get

$$\begin{aligned}(|u_t^n|^2+|\nabla u^n|^2+\|u^n\|_{L^p(\Omega)}^p)-C_4(1+|g|^2)&\le E\big(u^n(t)\big)\\&\le\beta'(|u_t^n|^2+|\nabla u^n|^2+\|u^n\|_{L^p(\Omega)}^p)+C_4(1+|g|^2),\end{aligned}\qquad(3.3)$$

where $C_4 = C_0' |\Omega| + \frac{1}{2\lambda_1}$, and $(\omega - \mu)|\nabla u_t^n|^2 \geq 2\alpha |u_t^n|^2$.

We put

$$A = -\alpha |u_t^n|^2 + \alpha |\nabla u^n|^2 + \alpha \langle f(u^n), u^n \rangle + \omega |\nabla u_t^n|^2 - \alpha (g, u^n).$$

Thus,

$$\mu |\nabla u_t^n|^2 + \alpha \beta'' (|u_t^n|^2 + |\nabla u^n|^2 + \| u^n \|_{L^p(\Omega)}^p) - \alpha C_0 |\Omega| - \frac{\alpha}{2\lambda_1} |g|^2 \leq A.$$

Then,

$$\frac{d}{dt} E\big(u^n(t)\big) + \mu |\nabla u_t^n|^2 + \gamma E\big(u^n(t)\big) \leq C_5 (1 + |g|^2). \tag{3.4}$$

Applying the Gronwall inequality to this estimate, we obtain

$$E\big(u^n(t)\big) \leq E\big(u^n(0)\big) e^{-\gamma t} + C_5 (1 + |g|^2)(1 - e^{-\gamma t}).$$

Using (3.3), we deduce

$$|u_t^n|^2 + |\nabla u^n|^2 + \| u^n \|_{L^p(\Omega)}^p \leq \frac{1}{\beta} E\big(u^n(0)\big) e^{-\gamma t} + \Big(\frac{C_5}{\beta} (1 - e^{-\gamma t}) + \frac{C_4}{\beta} \Big)(1 + |g|^2).$$

(3.5)

This estimate ensures that the solution $u^n(t)$ of (3.5) can be extended to $+\infty$.

And (3.4) gives

$$\frac{d}{dt} E\big(u^n(t)\big) + \mu |\nabla u_t^n|^2 \leq (C_5 + \gamma C_4)(1 + |g|^2).$$

Let T be an arbitrary positive number, integrating two sides of the above inequality from 0 to T , we get

$$E\big(u^n(t)\big) + \mu \int_0^T |\nabla u_t^n|^2 \, dt \leq E\big(u^n(0)\big) + C_6 T,$$

where, $C_6 = (C_5 + \gamma C_4)(1 + \| g \|^2)$.

Then,

$$|u_t^n|^2 + |\nabla u^n|^2 + \| u^n \|_{L^p(\Omega)}^p + \mu \int_0^T |\nabla u_t^n|^2 \, dt \leq C(\| (u^n(0), u_t^n(0)) \|_{\varepsilon(p)}, |g|^2).$$

This inequality yields

- $\{u^n\}_n$ is bounded in $L^\infty(0,T;H_0^1(\Omega)\cap L^p(\Omega))$;
- $\{u_t^n\}_n$ is bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$.

We first use the boundedness of $\{u^n\}_n$ in $L^\infty(0,T;L^p(\Omega))$ to prove the boundedness of $\{f(u^n)\}_n$ in $L^{p'}(\Omega_T)$, where p' is the conjugate of p . Indeed, the condition (1.2) implies that $|f(u)|\le C_7(1+|u|^{p-1})$.

Therefore,

$$\| f(u^n)\|_{L^{p'}(\Omega)}^{p'} = \int_0^T\int_\Omega |f(u^n)|^{p'}\,dxdt$$
$$\le C\int_0^T\int_\Omega (1+|u^n|^{(p-1)})^{p'}dxdt \le C\int_0^T\int_\Omega (1+|u^n|^p)dxdt.$$

Hence $\{f(u^n)\}$ is bounded in $L^{p'}(\Omega_T)$.

Next, we show that $\{u_{tt}^n\}_n$ is bounded in $L^{p'}(0,T;H^{-1}(\Omega)+L^{p'}(\Omega))$. Indeed, since $u_{tt}^n = g+\omega\Delta u_t^n + \Delta u^n - f(u^n)$,

we conclude that u_{tt}^n is bounded in $L^2(0,T;H^{-1}(\Omega)+L^{p'}(\Omega))$.

Combining with the fact that $L^2(0,T;H^{-1}(\Omega))$ and $L^{p'}(\Omega_T)$ are continuously embedded into $L^{p'}\left(0,T;H^{-1}(\Omega)+L^{p'}(\Omega)\right)$, we obtain the desired result.

From the above results, we can assume that

- $u^n \to u$ weakly in $L^\infty(0,T;H_0^1(\Omega)\cap L^p(\Omega))$;
- $u_t^n \to u_t$ weakly in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$;
- $u_{tt}^n \to u_{tt}$ weakly in $L^{p'}\left(0,T;H^{-1}(\Omega)+L^{p'}(\Omega)\right)$;
- $f(u^n)\to \chi$ weakly in $L^{p'}(\Omega_T)$.

It remains to show that $\chi = f(u)$, $u(0)=u_0$ and $u_t(0)=u_1$. Since $\{u^n\}_n$ is bounded in $L^2(0,T;H_0^1(\Omega)\cap L^p(\Omega))$ and $\{u_t^n\}_n$ is bounded in $L^{p'}\left(0,T;H^{-1}(\Omega)+L^{p'}(\Omega)\right)$, we deduce

$$u^n \to u \quad \text{in } L^2(0,T;L^2(\Omega)).$$

Hence, we can choose a subsequence $\{u^{n_k}\}$ such that

$$u^{n_k} \to u \quad \text{a.e in } \Omega_T.$$

It follows from the continuity of the function f that

$$f(u^{n_k}) \to f(u) \quad \text{a.e. in } \Omega_T.$$

In view of the boundedness of $\{f(u^{n_k})\}_k$ in $L^{p'}(\Omega_T)$, by Lemma 1.3 in $[7, Chapter\ 1]$, we have

$$f(u^{n_k}) \to f(u) \quad \text{waekly in} \quad L^{p'}(\Omega_T),$$

and taking into account the uniqueness of a weak limit, we get $\chi = f(u)$.

To prove $u(0) = u_0, u_t(0) = u_1$, choosing some test functio $\varphi \in C^2\left([0,T]; H_0^1(\Omega) \cap L^p(\Omega)\right)$ with $\varphi(T) = 0, \varphi_t(T) = 0$ and integrating by parts in t in the approximate equations, we have

$$\int_0^T -\langle u_t^n, \varphi' \rangle dt + \int_{\Omega_T} \left(\omega \nabla u_t^n \nabla \varphi + \nabla u^n \nabla \varphi + f(u^n)\varphi - g\varphi\right) dxdt = (u_t^n(0), \varphi(0)).$$

Taking limits as $n \to \infty$ we obtain

$$\int_0^T -\langle u_t, \varphi' \rangle dt + \int_{\Omega_T} \left(\omega \nabla u_t \nabla \varphi + \nabla u \nabla \varphi + f(u)\varphi - g\varphi\right) dxdt = (u_1, \varphi(0)),$$

since $u_t^n(0) \to u_1$.

On the other hand, for the "*limiting equation*", we have

$$\int_0^T -\langle u_t, \varphi' \rangle dt + \int_{\Omega_T} \left(\omega \nabla u_t \nabla \varphi + \nabla u \nabla \varphi + f(u)\varphi - g\varphi\right) dxdt = (u_t(0), \varphi(0)).$$

(3.6)

Comparing (3.6) with (3.6) we get $u_t(0) = u_1$.

By the same way, we obtain

$$\int_0^T \langle u^n, \varphi'' \rangle dt + \int_{\Omega_T} \left(\omega \nabla u_t^n \nabla \varphi + \nabla u^n \nabla \varphi + f(u^n)\varphi - g\varphi \right) dxdt$$
$$= (u_t^n(0), \varphi(0)) - (u^n(0), \varphi_t(0)).$$

Taking limits as $n \to \infty$ we obtain

$$\int_0^T \langle u, \varphi'' \rangle dt + \int_{\Omega_T} \left(\omega \nabla u_t \nabla \varphi + \nabla u \nabla \varphi + f(u)\varphi - g\varphi \right) dxdt \quad (3.7)$$
$$= (u_1, \varphi(0)) - (u_0, \varphi_t(0)),$$

since $u^n(0) \to u_0$.

On the other hand, for the "*limiting equation*", we have

$$\int_0^T \langle u, \varphi'' \rangle dt + \int_{\Omega_T} \left(\omega \nabla u_t \nabla \varphi + \nabla u \nabla \varphi + f(u)\varphi - g\varphi \right) dxdt$$
$$= (u_1, \varphi(0)) - (u(0), \varphi_t(0)). \quad (3.8)$$

Comparing (3.7) with (3.8) we get $u(0) = u_0$.

Thus, $u(x,t)$ is a weak solution to (1.1). The global existence of the solution $u(x,t)$ follows from the following inequality, which is proved similarly to (3.5),

$$|u_t|^2 + |\nabla u|^2 + \|u\|_{L^p(\Omega)}^p \leq C(\|(u_0,u_1)\|_{\varepsilon(p)})e^{-\gamma t} + \left(\frac{C_5}{\beta}(1 - e^{-\gamma t}) + \frac{C_4}{\beta} \right)(1 + |g|^2).$$

(ii) Uniqueness and continuous dependence. Let u, v be two solutions of problem (1.1) with initial data $u_0, v_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$. Then $w = u - v$ satisfies

$$\begin{cases} w_{tt} - \omega \Delta w_t - \Delta w + f(u) - f(v) = 0, \; x \in \Omega, t > 0 \\ w|_{\partial\Omega} = 0, \; w(0) = u_0 - v_0, \; w_t(0) = u_1 - v_1. \end{cases} \quad (3.9)$$

For function $\theta(x) \in [0,1]$, since (1.2), we get

$$(f(u) - f(v))(u - v) = f'(\eta)(u - v)^2 \geq -C_3 |w|^2 + C_3 |\theta u + (1-\theta)v|^{p-2} |w|^2$$

and

$$|f(u) - f(v)| \leq C_3(1 + |\theta u + (1-\theta)v|^{p-2})|w| \leq C_3 |w| + C_3 |\theta u + (1-\theta)v|^{p-2} |w|.$$

Multiplying Eq. (3.9) by αw and $-\Delta^{-1} w_t$, respectively, then integrating over Ω, we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}\alpha\omega|\nabla w|^2+\alpha\langle w_t,w\rangle\Big)+\alpha|\nabla w|^2+\alpha C_3\langle|\theta u+(1-\theta)v|^{p-2},\|w\|^2\rangle$$
$$\le \alpha C_3|w|^2+\alpha\|w_t\|^2$$

(3.10)

and

$$\frac{d}{dt}\Big(|\Delta^{-1/2}w_t|^2+|w|^2\Big)+2\omega|w_t|^2\le 2\langle|f(u)-f(v)|,|-\Delta^{-1}w_t|\rangle.$$

(3.11)

Using the growth assumptions on f and the embedding $H^{2-s}(\Omega)\to C(\Omega)$ for $s\in[0;1/2)$ yield

$$\|\Delta^{-1}w_t\|_{C(\Omega)}\le\|\Delta^{-1}w_t\|_{H^{2-s}(\Omega)}\le\|w_t\|_{H^{-1}(\Omega)}$$

and

$$2\langle|f(u)-f(v)|,|-\Delta^{-1}w_t|\rangle\le\varepsilon'\|f(u)-f(v)\|^2_{L^1(\Omega)}+\frac{1}{\varepsilon'}\|w_t\|^2_{H^{-1}(\Omega)}.$$

Moreover,

$$\begin{aligned}\|f(u)-f(v)\|^2_{L^1}&\le C|w|^2+C\langle|\theta u+(1-\theta)v|^{p-2},|w|\rangle^2\\&\le C|w|^2+C\langle|\theta u+(1-\theta)v|^{p-2},1\rangle\langle|\theta u+(1-\theta)v|^{p-2},|w|^2\rangle\\&\le C|w|^2+C'\langle|\theta u+(1-\theta)v|^{p-2},|w|^2\rangle.\end{aligned}$$

Combining (3.10) with (3.11), we deduce

$$\frac{d}{dt}\Big(\frac{1}{2}\alpha\omega|\nabla w|^2+\alpha\langle w_t,w\rangle+\|w_t\|^2_{H^{-1}(\Omega)}+|w|^2\Big)+\alpha|\nabla w|^2+(2\omega-\alpha)|w_t|^2$$
$$\le C|w|^2+\frac{1}{\varepsilon'}\|w_t\|_{H^{-1}(\Omega)}.$$

Putting

$$E_{-1}(w)=\frac{1}{2}\alpha\omega|\nabla w|^2+\alpha\langle w_t,w\rangle+\|w_t\|^2_{H^{-1}(\Omega)}+|w|^2.$$

Choose α,ε small enough, then

$$K_1(\|w\|^2_{H^1(\Omega)}+\|w_t\|^2_{H^{-1}(\Omega)})\le E_{-1}(w)\le K_2(\|w\|^2_{H^1(\Omega)}+\|w_t\|^2_{H^{-1}(\Omega)}).$$

Thus

$$\frac{d}{dt}E_{-1}(w)+C(|\nabla w|^2+|w_t|^2)\le C(|w|^2+\|w_t\|^2_{H^{-1}(\Omega)})\le KE_{-1}(w).$$

Applying Gronwall inequality, we get

$$E_{-1}(w)\le E_{-1}(w(0))e^{Kt},$$

and

$$\|w(t)\|^2_{H^1(\Omega)}+\|w_t(t)\|^2_{H^{-1}(\Omega)}\le Ce^{Kt}(\|w(0)\|^2_{H^1(\Omega)}+\|w_t(0)\|^2_{H^{-1}(\Omega)}).$$

This implies the uniqueness if $(u_0=v_0,u_1=v_1)$ and the continuous dependence of the solution with respect to the initial data.

4. WEAK GLOBAL ATTRACTOR

In this section, we will construct the global attractor for the semigroup $S(t)$ generated by weak energy solutions of (1.1). Unfortunately, we are not able to construct the global attractor in a strong topology of $\varepsilon(p)$, but only in the weak topology of $\varepsilon_{-1}:=L^2(\Omega)\times H^{-1}(\Omega)$. Namely, a set $A\subset\varepsilon(p)$ is a weak global attractor of a solution semigroup $S(t)$: $\varepsilon(p)\to\varepsilon(p)$ generated by Eq. (1.1) (the so-called $(\varepsilon(p),\ \varepsilon_{-1})$-attractor in the terminology of Babin and Vishik, see $[1]$) if

(1) The set A is bounded in $\varepsilon(p)$ and compact in ε^{-1}

(2) A is an invariant sets, i.e., $S(t)A=A, t\ge 0;$

(3) A attracts all bounded sets of $\varepsilon(p)$ in the topology of ε_{-1}. That is, for any bounded subset B of $\varepsilon(p)$,

$$\delta_{\varepsilon_{-1}}(S(t)B,A)\to 0 \text{ as } t\to\infty.$$

From (3.5), we deduce the existence of an absorbing set in $\varepsilon(p)$, that is, there are a constant R and a time $t_0(u_0,u_1)$ such that for the solution $(u(t),u_t(t))=S(t)(u_0,u_1)$, we have

$$|u_t|^2+|\nabla u|^2+\|u\|^p_{L^p(\Omega)}\le R \text{ for all } t\ge t(u_0,u_1).$$

It follows the ball B_0 centered at 0 with radius R is an absorbing set for $S(t)$ in $\varepsilon(p)$, and noting that $\varepsilon(p) \to \varepsilon_{-1}$ compactively and ε_{-1} is connected, we obtain the following theorem.

Theorem 4.1. Under condition (1.2), the semigroup $S(t)$ associated to problem (1.1) possesses a weak global attractor $\mathrm{A} = \omega(B_0)$ in ε_{-1} .

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