MASSERA -TYPE THEOREM FOR DIFFERENTIAL EQUATIONS IN THE SPACE OF BOUNDED CONTINUOUS FUNCTIONS

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Abstract: Consider the class of the differential equations in the form

$$\frac{\partial}{\partial t}\big(x(t)+h(t)\big)=B(t)x(t)+k(t),\ t\in\mathbb{R}_+. \qquad (1)$$

By using Massera-type theorem, we demonstrate the existence and uniqueness of periodic solutions for the equation (1), with the operator $B(t): X \supseteq D(B) \rightarrow X$ is α-periodic and possibly unbounded on a Banach space X; the function h gets the value in the Banach space D(B), the function k attains the value in the Banach space X and is α-periodic.

Keywords: Massera-type theorem, Differential equation, periodic solution.

ĐỊNH LÝ DẠNG MASSERA CHO PHƯƠNG TRÌNH VI PHÂN TRONG KHÔNG GIAN CÁC HÀM LIÊN TỤC BỊ CHẶN

Tóm tắt: Xét lớp các phương trình vi phân có dạng

$$\frac{\partial}{\partial t}\big(x(t)+h(t)\big)=B(t)x(t)+k(t),\ t\in\mathbb{R}_+. \qquad (1)$$

Bằng việc sử dụng đinh lý dạng Massera, chúng tôi chứng minh được sự tồn tại và duy nhất nghiệm tuần hoàn của phương trình (1), với toán tử $B(t): X \supseteq D(B) \rightarrow X$ tuần hoàn chu kì α và có thể không bị chăn trên không gian Banach X; hàm nhận giá trị trong không gian Banach D(B), hàm k nhận giá trị trong không gian Banach X và tuần hoàn chu kì α.

Từ khóa: Định lý dạng Masera, Phương trình vi phân, nghiệm tuần hoàn.

1. Introdution

Generally, for differential equations, there are several methods used to find conditions for the existence, uniqueness of a periodic solution. Such as Massera's principle[1], special fixed-point methods, e.g., Hale [2], Ezzinbi [4], …. The most popular approaches used in this direction are the ultimate boundedness of solution and the compactness of Poincare' map realized through some compact embeddings. However, if unbounded domains or equations that have unbounded solutions, such compact embeddings do not hold, and the choice of appropriate initial vectors is not easy.

One may invoke to the so-called Massera-type theorem to overcome such difficulties. That theorem roughly saying that if a differential equation has a bounded solution then it has a periodic one. Actually, in Ha, Huy & Loan [5], this methodology has been invoked to prove the existence and uniqueness of a periodic solution for the case of partial neutral functional differential.

In our research, by Massera-type theorem, the existence and uniqueness of periodic solutions to partial differential equations in the spaces of bounded continuous functions are considered. Here, the function h get valid in a Banach space D(B), k taking values in Banach space X. The difficulties we face is express the formula for the mild solution of equation (*). To overcome such difficulties, we establish the assumption operator B satisfies the Standing Hypothesis 2.4 (it will be presented below). My main result is Theorem 2.1.

2. Literature review, Theoretical framework and Methodology

One of the important research directions related to the asymptotic behavior of the solutions to the above equation is to find conditions for the existence, uniqueness of a periodic solution and conditional stability of periodic solution to the equation. Many other authors (see [1],[2],[4] and references therein) have contributed to the development of the theory for this type of problem.

The authors have studied the qualitative properties of this problem obtaining important results on asymptotic behavior of the solutions around the stationary ones or along specific trajectory.

There have been some useful methods for differential equations in that direction such as specific fixed point theorems (see Hale [2]) or Massera's principle (see Huy [5]). In many circumstances, the popular approaches are the ultimate boundedness of solutions and the Poincar'e map realized through some compact embeddings (see, Benkhalti [4]). However, in many specific applications, e.g., for the partial differential equations in unbounded domains or equations possessing unbounded solutions, such compact embeddings do not hold, and it is not easy to choose an appropriate initial vector from which emanating a bounded solution. One way to overcome such difficulties is invoking to Massera-typed roughly saying that if a differential equation has a bounded solution then it has a periodic one. In Huy [5] this Massera-type theorem was invoked in combination with Admissible spaces to establish the existence of periodic solutions to Partial Neutral Functional Differential Equations.

3. Massera-typed theorem to partial differential equations in the spaces of bounded continuous functions

For a Banach space X, we investigate the following partial differential equation in the form

$$\begin{cases} \frac{\partial}{\partial t}\left(x(t)+h(t)\right)=B(t)x(t)+k(t),\ t\in\mathbb{R}_+ \\ x(0)=\phi\in D(B), \end{cases} \tag{2}$$

where, D(B) ⊂ X, the function k taking values in Banach space X, h get valid in a Banach space D(B). The operator $B(t)$ is a linear operator on a Banach space *D(B)* such that the Cauchy problem

$$\begin{cases} \frac{dx}{dt}=B(t)x(t),\ t\geq s\geq 0 \\ x(s)=u\in X, \end{cases} \tag{3}$$

is well-posed, there exist an evolution family $(U(t,s))_{t\geq s\geq 0}$ such that the solution of the Cauchy problem (3) is given by $x(t)=U(t,s)x(s)$.

Definition 2.1. (see Pazy [3]) A family of bounded linear operator $(U(t,s))_{t\geq s\geq 0}$ on a Banach space X is a evolution family if:

i) The map $(t,s)\mapsto U(t,s)u$ is continuous for every $u\in X$, where $(t,s)\in\{(t,s)\in\mathbb{R}^2 : t\geq s\geq 0\}$;

ii) $U(t,t)=Id, U(t,r)U(r,s)=U(t,s)$ for all $t\geq r\geq s\geq 0$;

iii)exists $H,\lambda\geq 0$ such that $\|U(t,s)u\|\leq He^{\lambda(t-s)}\|u\|, \forall t\geq s\geq 0, u\in X.$

Next, some preliminaries are recalled for late uses. That are the spaces of bounded continuous functions is defined as

$$M=BC(\mathbb{R}_+,X):=\{y:\mathbb{R}_+\to X \mid y \text{ is continuous and } \sup_{t\in\mathbb{R}_+}\|y(t)\|<\infty\},$$

endowed with the norm $\|y\|_M:=\sup_{t\in\mathbb{R}_+}\|y(t)\|$; and, the Banach space

$$\mathfrak{M} = BC(\mathbb{R}_+, D(B)) = \{y : \mathbb{R}_+ \to D(B) \mid y \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \| y(t) \|_{D(B)} < \infty\},$$

endowed with the norm $\| y \|_{\mathfrak{M}} := \sup_{t\in\mathbb{R}_+} \| y(t) \|_{D(B)}$.

Assumption 2.2. Suppose that $B(t+\alpha) = B(t)$, it mean $B(t)$ is α-periodic, for a fixed constant $\alpha > 0$ and for all $t \in \mathbb{R}_+$, then $(U(t,s))_{t\geq s\geq 0}$ become α-periodic in the sense that $U(t+\alpha, s+\alpha) = U(t,s),\ \forall\ t \geq s \geq 0.$

Assumption 2.3. We suppose that the Banach space D(B)$\subset X$ possesses a separable pre-dual Y, (i.e., *D(B) = Y'),* where *Y* is a separable Banach space. Assume that the space *Y* considered as a subspace of *Y''* (through the canonical embedding) is invariant under the operator *U' (*α*,* 0*)* which is the dual of *U(*α*,* 0*).*

Standing Hypothesis 2.4. We assume the domain of each B(t) is independent of t, and denoted by D(B) which is a Banach space with norm $\|\cdot\|_{D(B)}$ such that $\| B(t)x \| \leq \chi(t) \| x \|_{D(B)}$, ∀ x ∈ D(B), $\chi(t)$ belongs to M. And the family of operators $(B(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ as defined in the Definition 2.1.

For conveniently represent the mild solution formula, we can now define the operator $\Phi(t) : \mathbb{R}_+ \to M$, with $\Phi(t) := -B(t)h(t) + k(t)$, then $\Phi(t)$ is α-periodic provided that $B(t)$, $h(t)$ and $k(t)$ are α-periodic. Since Standing hypothesis 2.4, we have

$$\begin{aligned}\|\Phi(t)\| &\leq \|-B(t)h(t)\| + \|k(t)\| \leq \chi(t)\|h(t)\|_{D(B)} + \|k(t)\| \\ &\leq |\chi(t)| \sup_{t\geq 0}\|h(t)\|_{D(B)} + \|k(t)\| \\ &\leq |\chi(t)|\|h\|_{\mathfrak{M}} + \|k(t)\|.\end{aligned}$$

Therefore

$$\|\Phi(t)\|_M = \operatorname*{Sup}_{t\geq 0}\|\Phi(t)\| \leq \|\alpha\|_{\mathfrak{M}}\|\chi\|_M + \|\beta(t)\|_M.$$

Then we rewrite the equation (2) in the form

$$\begin{cases} \frac{\partial}{\partial t}\big(x(t)+h(t)\big) = B(t)\big(x(t)+h(t)\big) + \Phi(t),\ t \in \mathbb{R}_+, \\ x(0) = \phi \in D(B). \end{cases} \quad (4)$$

A function *x* satisfying the following integral equation

$$x(t) + h(t) = U(t,0)\big(\phi + h(0)\big) + \int_0^t U(t,\tau)\Phi(\tau)d\tau,\ t \geq 0, \quad (5)$$

is called mild solution to (4). It means

$$x(t) = -h(t) + U(t,0)\big(\phi + h(0)\big) + \int_0^t U(t,\tau)\Phi(\tau)d\tau \text{, for all } t \geq 0. \qquad (6)$$

The existence and uniqueness of the periodic solution of equation (4) is given in the following theorem.

Theorem 2.1. *For the Banach space X,* possesses a separable pre-dual *Y, D(B) ⊂X. Under Standing Hypothesis 2.4 and assumption 2.2 and Assumption 2.3 are satisfied. With constant E > 0, there exists $\phi \in$ D(B) such that the mild solution x of equation (4) with $x(0) = \phi$ satisfies $x \in \mathfrak{M}$ and*

$$\| x \|_{\mathfrak{M}} \leq (E\|\chi\|_M + \alpha) \| h \|_{\mathfrak{M}} + E \| k \|_M . \qquad (7)$$

Furthermore, if the evolution family $\big(U(t,s)\big)_{t\geq s\geq 0}$ satisfies $\lim_{t\to\infty} \| U(t,0)x \| = 0$ for $x \in X$ such that U(t,0)x is bounded in $\mathbb{R}_+$. Then the equation (3) has unique α-periodic solution $\hat{x}$ satisfies

$$\| \hat{x} \|_{\mathfrak{M}} \leq \Big((E+\alpha)Ke^{\lambda\alpha}\|\chi\|_M + Ke^{\lambda\alpha} + \alpha\Big) \| h \|_{\mathfrak{M}} + (E+\alpha)Ke^{\lambda\alpha} \| k \|_M . \qquad (8)$$

Proof. This theorem is proved through following three steps. The first, we will prove that the equation (4) has α-periodic solution $\hat{x}$. To do this, we prove that if *x(t)* is mild solution of (4) then *x(t+nα)* is also mild solution to this equation. Indeed, from evolution family's property and the mild solution (5), we have

$$x(t+n\alpha) + h(t+n\alpha)$$

$$= U(t+n\alpha,0)[\phi + h(0)] + \int_0^{t+n\alpha} U(t+n\alpha,T)\Phi(T)dT$$

$$= U(t,0)U(n\alpha,0)[\phi + h(0)] + \int_0^{n\alpha} U(t,0)U(n\alpha,T)\Phi(T)dT + \int_0^t U(t,T)\Phi(T)dT$$

$$= U(t,0)[x(n\alpha) + h(n\alpha)] + \int_0^t U(t,T)\Phi(T)dT.$$

Since $h(t)$ is α-periodic then $h(n\alpha) = h(0)$, we obtain that

$$x(t+n\alpha) + h(t+n\alpha) = U(t,0)[x(n\alpha) + h(0)] + \int_0^t U(t,T)\Phi(T)dT.$$

It mean, *x(t+nα)* is mild solution of equation (3). We next let *t=*α be obtained

$$x\left((n+1)\alpha\right)+h\left((n+1)\alpha\right)=U(\alpha,0)[x(n\alpha)+h(0)]+\int_0^{\alpha}U(\alpha,T)\Phi(T)dT. \quad (9)$$

For each $n\in\mathbb{N}$, we define the Ces`aro sum y_m by

$$y_m:=\frac{1}{m}\sum_{n=1}^{m}[x(n\alpha)+h(0)].$$

The estimate (7) implies

$$\sup_{n\in\mathbb{N}}\|x(n\alpha)\|_{D(B)}\le(E\|\chi\|_M+\alpha)\|h\|_{\mathfrak{M}}+E\|k\|_M.$$

Therefore, the sequence $\{y_m\}_{m\in\mathbb{N}}$ is bounded in *D(B)* and

$$\sup_{m\in\mathbb{N}}\|y\|_{D(B)}\le h(0)+(E\|\chi\|_M+\alpha)\|h\|_{\mathfrak{M}}+E\|k\|_M.$$

According to the Standing Hypothesis 2.4, the Banach space *D(B)=Y'* and *Y* is separable by Banach-Alaoglu's Theorem there exists a subsequence $\{y_{m_j}\}$ of $\{y_m\}$ such that

$$\{y_{m_j}\}\xrightarrow{weak*}y^*\in D(B); \quad (10)$$

with $\|y*\|_{\mathfrak{M}}\le h(0)+(E\|\chi\|_M+\alpha)\|h\|_{\mathfrak{M}}+E\|k\|_M$. Other way, from (9) we have.

$$U(\alpha,0)[x(n\alpha)+h(n\alpha)]=x\left((n+1)\alpha\right)+h\left((n+1)\alpha\right)-\int_0^{\alpha}U(t,T)\Phi(T)dT$$

$$=h(0)+x\left((n+1)\alpha\right)-\int_0^{\alpha}U(t,T)\Phi(T)dT.$$

Then

$$U(\alpha,0)\sum_{n=1}^{m}[x(n\alpha)+h(0)]=\sum_{n=1}^{m}[x(n\alpha)+h(0)]-m\int_0^{\alpha}\left(U(\alpha,T)\Phi(T)-x(\alpha)-x\left((m+1)\alpha\right)\right)dT.$$

It means

$$U(\alpha,0)y_m+\int_0^{\alpha}U(\alpha,T)\Phi(\tau)d\tau-y_m=\frac{1}{m}\left(U((m+1)\alpha)-U(\alpha)\right).$$

Since the sequence $\{x(m)\}_{m\in\mathbb{N}}$ is bounded in *D(B)*, we obtain that

$$\lim_{m\to\infty}\frac{1}{m}\left(x((m+1)\alpha)-x(\alpha)\right)=0,$$

strongly in *D(B)*. Therefore,

$$U(\alpha,0)y_m + \int_0^\alpha U(\alpha,T)\Phi(T)dT - y_m \overset{m\to\infty}{\to} 0. \quad (11)$$

From (11) and the sequence $\{y_{m_j}\}$ satisfies (10) we obtain that

$$U(\alpha,0)y_{m_j} + \int_0^\alpha U(\alpha,T)\Phi(T)dT \overset{weak*}{\to} y^* \in D(B). \quad (12)$$

Next, we will prove that

$$U(\alpha,0)y^* + \int_0^\alpha U(\alpha,T)\Phi(T)dT = y^*. \quad (13)$$

Indeed, denoting by $\langle\cdot,\cdot\rangle$ the dual pair between *Y* and *Y'*, using the fact that *U'(T,0)* leaves *Y* invariant for all $z \in Y$, we have

$$\langle U(\alpha,0)y_{m_j} + \int_0^\alpha U(\alpha,T)\Phi(T)dT, z\rangle$$

$$= \langle U(\alpha,0)y_{m_j}, z\rangle + \langle \int_0^\alpha U(\alpha,T)\Phi(T)dT, z\rangle$$

$$= \left\langle y_{m_j}, U'(\alpha,0)z\right\rangle + \left\langle \int_0^\alpha U(\alpha,T)\Phi(T)dT, z\right\rangle \overset{m_j\to\infty}{\to} \left\langle y^*, U'(\alpha,0)z\right\rangle + \left\langle \int_0^\alpha U(\alpha,T)\Phi(T)dT, z\right\rangle$$

$$= \left\langle U(\alpha,0)y^*, z\rangle + \langle \int_0^\alpha U(\alpha,T)\Phi(T)dT, z\right\rangle$$

$$= \left\langle U(\alpha,0)y^* + \int_0^\alpha U(\alpha,T)\Phi(T)dT, z\right\rangle.$$

Hence,

$$U(\alpha,0)y_{m_j} + \int_0^\alpha U(\alpha,T)\Phi(T)dT \overset{weak*}{\to} U(\alpha,0)y^* + \int_0^\alpha U(\alpha,T)\Phi(T)dT. \quad (14)$$

It now follows from (12), (14) and due to the uniqueness of the limit that (13). This yields, the mild solution $x^* \in \mathfrak{M}$ of equation (4) with the initial value $x^*(0) = y^* - h(0)$ satisfies

$$x^*(\alpha) + h(0) = y^* = x^*(0) + h(0).$$

This is equivalent to *x*(*α*)=x*(0)*. It means, *x*(t)* is α -periodic.

The second, we prove inequality (8). Since mild solution *x*(t)* of equation (4) is α -periodic, we have

$$\sup_{t\in\mathbb{R}_+} \| x^*(t) \|_{D(B)} = \sup_{t\in[0,\alpha]} \| x^*(t) \|_{D(B)}.$$

From (7), we obtain that

$$\sup_{t\in\mathbb{R}_+} \| x^*(t) \|_{D(B)} = \sup_{t\in[0,\alpha]} \| U(\alpha,0)y^* + \int_0^t U(t,T)\Phi(T)dT - h(t) \|$$

$$\leq \sup_{t\in[0,\alpha]} \| U(t,0)y^* + \int_0^t U(t,T)\Phi(T)dT \| + \sup_{t\in[0,\alpha]} \| h(t) \|_{D(B)}$$

$$\leq Ke^{\lambda\alpha} \| y^* \|_{\mathfrak{M}} + Ke^{\lambda\alpha} \| \Phi \|_M + \| h \|_{\mathfrak{M}}$$

$$\leq (E+\alpha)Ke^{\lambda\alpha} \| \Phi \|_M + (Ke^{\lambda\alpha} + \alpha) \| h \|_{\mathfrak{M}}$$

$$\leq \left((E+\alpha)Ke^{\lambda\alpha} \|\chi\| + Ke^{\lambda\alpha} + \alpha \right) \| h \|_{\mathfrak{M}} + (E+\alpha)Ke^{\lambda\alpha} \| k \|_M .$$

The last step, we proof the α-periodic mild solution of equation (4) is unique. Let x_1^* and x_2^* be two α-periodic mild solutions to equation (4). It mean

$$x_1^*(t) = -h(t) + U(t,0)[\phi_1 + h(0)] + \int_0^t U(t,T)\Phi(T)dT, t \geq 0;$$

$$x_2^*(t) = -h(t) + U(t,0)[\phi_2 + h(0)] + \int_0^t U(t,T)\Phi(T)dT, t \geq 0.$$

Setting $v = x_1^* - x_2^*$, then we have that v is α-periodic and we obtain that

$$v(t) = U(t,0)\left(\phi_1 - \phi_2\right), t \geq 0.$$

For $v(\cdot)$ is bounded on $\mathbb{R}_+$ and from their hypothesis of evolution family we have

$$\lim_{t\to\infty} \| v(t) \| = \lim_{t\to\infty} \| U(t,0)(\phi_1 - \phi_2) \| = 0 .$$

Combining with the periodicity of v, implies that v(t)=0 for $t \geq 0$. It means, $x_1^* = x_2^*$ Then, the periodic mild solution is unique. □

4. Conclusion

Our research established necessary conditions for linear partial differential equation (4) to have α-periodic mild solution, bounded and unique by using the Massera-type, roughly saying that if a differential equation has a bounded solution then it has a periodic one, in the spaces of bounded continuous functions. However, the stability and existence of manifolds around such a periodic solution have yet to be established. This is one of the areas where more research could be conducted. These issues, on the other hand, can be expanded to semi-linear equations.

Our study of this equation will be a stepping stone to developing semi-linear differential equations, where the functions h depend on the state variable. This class of equations can appear in nonlinear oscillation phenomena (see Hernández [6]), or arise from population dynamics (see Benkhalti [4], Wu [8]).

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