SYMMETRIZED *(p,h)*-CONVEXITY AND SOME HERMITE-HADAMARD-TYPE INEQUALITIES

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SUMMARY

This paper introduces symmetrized *(p,h)*-convex functions and establishes some Hermite-Hadamard-type inequalities for the new class of functions.

Keywords: *symmetrized (p,h)-convexity, (p,h)-convexity, p-convexity, h-convexity, harmonically convexity, Hermite-Hadamard inequality.*

1. INTRODUCTION

A function f is said to be *convex* on a real interval I if

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y),$$

for all $x,y\in I$ and $\lambda\in[0,1]$. An estimate of the (integral) mean value of a continuous convex function is known as Hermite-Hadamard inequality (Hadamard, 1893, Hermite, 1883). Precisely, if f is a convex function on a real interval I and $a,b\in I$ with $a<b$ then

$$f\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b f(x)dx\le \frac{f(a)+f(b)}{2}.$$

This inequality is one of the most useful inequalities in mathematical analysis. For new proofs, noteworthy extensions, generalizations, and numerous applications of this inequality, see e.g., (Dragomir & Pearce, 2003, Mitrinović & Lacković, 1985).

In recent years, convexity has been generalized and extended in various aspects using new and different concepts. These researches led to the appearance of several Hermite-Hadamard-type inequalities. Among these generalizations, two interesting concepts are *p*-convex functions (Zhang & Wan, 2007) and *h*-convex functions (Varošanec, 2007). They are defined as follows:

Definition 1.1 (Zhang & Wan, 2007). Let $I\subset(0,\infty)$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. We say that $f:I\to\mathbb{R}$ is a *p*-convex function if

$$f\left([\alpha x^p+(1-\alpha)y^p]^{1/p}\right)\le \alpha f(x)+(1-\alpha)f(y),$$

for all $x,y\in I$ and $\alpha\in[0,1]$.

Definition 1.2 (Varošanec, 2007). Let I,J be two real intervals with $J\supseteq(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative and non-zero function. We say that $f:I\to\mathbb{R}$ is a *h*-convex function if f is non-negative and

$$f(\alpha x+(1-\alpha)y)\le h(\alpha)f(x)+h(1-\alpha)f(y),$$

for all $x,y\in I$ and $\alpha\in(0,1)$.

These functions are continuously generalized to (*p,h*)-convex functions by Fang & Shi (2014):

Definition 1.3 (Fang & Shi, 2014). Let $I\subset(0,\infty)$ and $J\supseteq(0,1)$ be two real intervals, $p\in\mathbb{R}\setminus\{0\}$, and let $h:J\to\mathbb{R}$ be a non-negative and non-zero function. We say that $f:I\to\mathbb{R}$ be a *(p,h)*-convex function if f is non-negative and

$$f\left([\alpha x^p+(1-\alpha)y^p]^{1/p}\right)$$
$$\le h(\alpha)f(x)+h(1-\alpha)f(y),$$

for all $x,y\in I$ and $\alpha\in(0,1)$.

In (Fang & Shi, 2014), the authors also derived Hermite-Hadamard-type inequalities for the class of *(p,h)*-convex functions as follows:

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\left[\frac{a+b}{2}\right]^{1/p}\right)\le\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)dx$$
$$\le (f(a)+f(b))\int_0^1 h(t)dt. \quad (1.1)$$

In particular, if $p=-1$, *(p,h)*-convex functions become harmonically *h*-convex functions which are studied firstly by Noor & et al. (2015).

A significant generalization of *h*-convex functions is symmetrized *h*-convex functions, which are introduced by Dragomir (2016). To give this concept, the authors considered a symmetrical transform $\overline{f}$ of f on $[a,b]$ which is defined by

$$\overline{f}(x):=\frac{1}{2}[f(x)+f(a+b-x)].$$

Then we say $f:[a,b]\to[0,\infty)$ is symmetrized *h*-convex function on $[a,b]$ if $\overline{f}$ is *h*-convex on $[a,b]$. Moreover, if h integrable on $[0,1]$ and f integrable on $[a,b]$, Hermite-Hadamard-type inequality for symmetrized *h*-convex function is given as follows (Dragomir, 2016):

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx$$
$$\le\left[f(a)+f(b)\right]\int_0^1 h(x)dx. \quad (1.2)$$

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Subsequently, by considering an other symmetrical transform $\breve{f}$ of f on $[a,b]$ which is defined by

$$\breve{f}(x) := \frac{1}{2}\left[f(x) + f\left(\frac{abx}{(a+b)x - ab} \right) \right],$$

Wu & et al. (2017) gave symmetrized harmonic convex functions. In addition, the authors also introduced symmetrized harmonic *h*-convex functions, which is the generalization of harmonically convex functions and *h*-convex functions. Furthermore, if h integrable on $[0,1]$ and f is symmetrized harmonic *h*-convex and integrable on $[a,b]$, then Hermite-Hadamard-type inequality for f is given as follows (Wu & et al., 2017):

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$

$$\le [f(a)+f(b)]\int_0^1 h(x)dx. \quad (1.3)$$

Recently, İşcan (2020) used the *p*-symmetrical transform $P_{(f;p)}(x)$ of f on $[a,b]$ which is defined by

$$P_{(f;p)}(x) := \frac{1}{2}\left[f(x) + f\left([a^p + b^p - x^p]^{1/p} \right) \right]$$

to define symmetrized *p*-convex functions, then they provided the following Hermite-Hadamard-type inequality for f which is symmetrized *p*-convex and integrable on $[a,b]$:

$$f\left(\left[\frac{a^p + b^p}{2} \right]^{1/p} \right) \le \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx$$

$$\le \frac{f(a)+f(b)}{2}.$$

Essentially, the symmetrized *h*-convexity can be formed as symmetrized *(p,h)*-convexity, where $p=1$, while the symmetrized harmonic *h*-convexity are in the case $p=-1$. The main purpose of this paper is study symmetrized *(p,h)*-convexity for the more general case, i.e. $p \ne 0$. After introducing the definition, we establish Hermite-Hadamard-type inequalities for the new class. Moreover, this class is a generalization from the previous versions so some results in (Dragomir, 2016, Wu & et al., 2017, İşcan, 2020) are special cases of our results.

2. RESEARCH CONTENTS & METHODS

2.1. Research contents

We introduce the concept of the symmetrized (*p,h*)-convex function.

We establish some Hermite-Hadamard type inequalities symmetrized *(p,h)*-convex functions.

2.2. Research method

Based on the theory of symmetrized *(p,h)*-convexity, we give the comparison to achieve the results.

We combine analytical techniques and comparative methods to derive properties and inequalities.

3. RESULTS AND DISCUSSIONS

In this section, we consider a,b are positive numbers, $p \ne 0$, J is real interval satisfies $J \supseteq (0,1)$ and the function $h: J \to [0,\infty)$ is not identical to 0.

For a function $f:[a,b] \to \mathbb{R}$, we recall that a p-symmetrical transform of f on the interval $[a,b]$ is denoted by $P_{(f;p)}$ and is defined by

$$P_{(f;p)}(x) = \frac{1}{2}\left[f(x) + f\left([a^p + b^p - x^p]^{1/p} \right) \right],$$

for all $x \in [a,b]$.

Remark 3.1. If f is (p,h)-convex on $[a,b]$, then $P_{(f;p)}$ is also (p,h)-convex on $[a,b]$. Indeed, it is easy to see that $P_{(f;p)}$ is non-negative and for all $x,y \in [a,b], \alpha \in (0,1)$, we have

$$P_{(f;p)}\left([tx^p + (1-t)y^p]^{1/p} \right)$$

$$= \frac{1}{2}\Big[f\left([tx^p + (1-t)y^p]^{1/p} \right)$$

$$+ f\left([a^p + b^p - tx^p - (1-t)y^p]^{1/p} \right) \Big]$$

$$= \frac{1}{2}\Big[f\left([tx^p + (1-t)y^p]^{1/p} \right) +$$

$$f\left([t(a^p + b^p - x^p) + (1-t)(a^p + b^p - y^p)]^{1/p} \right) \Big]$$

$$\le \frac{1}{2}h(\alpha)\left[f(x) + f\left([a^p + b^p - x^p]^{1/p} \right) \right] +$$

$$\frac{1}{2}h(1-\alpha)\left[f(y) + f\left([a^p + b^p - y^p]^{1/p} \right) \right]$$

$$= h(\alpha)P_{(f;p)}(x) + h(1-\alpha)P_{(f;p)}(y).$$

However, if $P_{(f;p)}$ is (p,h)-convex on $[a,b]$ for a function $f:[a,b] \to \mathbb{R}$, then the function f is not necessarily (p,h)-convex on $[a,b]$. For example, let $h(\alpha) = \alpha,\ \alpha \in (0,1)$ and $p=-1$, consider $f(x) = -\ln x, x \in (0,1)$. The function f is not (p,h)-convex but $P_{(f;-1)}$ is (p,h)-convex (Wu. & et al. 2017).

Definition 3.2. A function $f:[a,b] \to \mathbb{R}$ is said to be symmetrized (p,h)-convex on $[a,b]$ if the *p*-symmetrical transform $P_{(f;p)}$ is (p,h)

-convex on $[a,b]$.

In other words, a function f is symmetrized (p,h)-convex on $[a,b]$ if

$$P_{(f;p)}\left([\alpha x^p+(1-\alpha)y^p]^{1/p}\right)$$
$$\le h(\alpha)P_{(f;p)}(x)+h(1-\alpha)P_{(f;p)}(y),$$

for all $x,y\in[a,b]$ and $\alpha\in(0,1)$.

By Remark 3.1, if f is (p,h)-convex on $[a,b]$, then f is symmetrized (p,h)-convex on $[a,b]$. Also, if $[c,d]\subset[a,b]$ and f symmetrized (p,h)-convex on $[a,b]$, this does not imply in general that f is symmetrized (p,h)-convex on $[c,d]$.

Example 3.3. Let $J\supseteq(0,1)$, and the function $h:J\to[0,\infty)$ satisfies $h(\alpha)\ge\alpha$, for all $\alpha\in(0,1)$, and $\lambda\ge 2$. Then the function $f:[a,b]\to\mathbb{R}$, $f(x)=(x^p-a^p)^{\lambda-1}, p>0$ is symmetrized (p,h)-convex on $[a,b]$.

Indeed, for any $u,v\in[a,b]$ and any $\alpha\in(0,1)$, by convexity of the function $g(\psi)=\psi^{\lambda-1},\psi\ge 0$, it is easily seen that

$$f\left([\alpha u^p+(1-\alpha)v^p]^{1/p}\right)$$
$$=(\alpha u^p+(1-\alpha)v^p-a^p)^{\lambda-1}$$
$$=(\alpha(u^p-a^p)+(1-\alpha)(v^p-a^p))^{\lambda-1}$$
$$\le\alpha(u^p-a^p)^{\lambda-1}+(1-\alpha)(v^p-a^p)^{\lambda-1}$$
$$\le h(\alpha)(u^p-a^p)^{\lambda-1}+h(1-\alpha)(v^p-a^p)^{\lambda-1}$$
$$=h(\alpha)f(u)+h(1-\alpha)f(v).$$

Thus $P_{(f;p)}$ is also (p,h)-convex on $[a,b]$. Therefore f is symmetrized (p,h)-convex on $[a,b]$.

Proposition 3.4. *Let $f:[a,b]\to\mathbb{R}$ and $g(x)=x^{1/p},x>0,p\ne 0$. A function f is symmetrized (p,h)-convex on $[a,b]$ if and only if $f\circ g$ is symmetrized h-convex on $g^{-1}([a,b])=[a^p,b^p]$ (or $[b^p,a^p]$).*

Proof. Let f is a symmetrized (p,h)-convex on $[a,b]$ and x,y $x,y\in g^{-1}([a,b])$ be arbitrary, then there exists $u,v\in[a,b]$ such that $x=u^p, y=v^p$.

$$\overline{f\circ g}\left(\alpha x+(1-\alpha)y\right)$$
$$=\frac{1}{2}\Big[(f\circ g)\left(\alpha x+(1-\alpha)y\right)$$
$$+(f\circ g)\left(a^p+b^p-\alpha x-(1-\alpha)y\right)\Big]$$
$$=\frac{1}{2}\Big[f\left([\alpha u^p+(1-\alpha)v^p]^{1/p}\right)$$
$$+f\left([a^p+b^p-\alpha u^p-(1-\alpha)v^p]^{1/p}\right)\Big]$$
$$=P_{(f;p)}\left([\alpha u^p+(1-\alpha)v^p]^{1/p}\right).$$

Since f is symmetrized (p,h)-convex on $[a,b]$, it is easily seen that

$$P_{(f;p)}\left([\alpha u^p+(1-\alpha)v^p]^{1/p}\right)$$
$$\le h(\alpha)P_{(f;p)}(u)+h(1-\alpha)P_{(f;p)}(v)$$
$$=\frac{1}{2}h(\alpha)\Big[f(u)+f\left([a^p+b^p-u^p]^{1/p}\right)\Big]$$
$$+\frac{1}{2}h(1-\alpha)\Big[f(v)+f\left([a^p+b^p-v^p]^{1/p}\right)\Big]$$
$$=\frac{1}{2}h(\alpha)\big[(f\circ g)(x)+(f\circ g)(a^p+b^p-x)\big]$$
$$+\frac{1}{2}h(1-\alpha)\big[(f\circ g)(y)+(f\circ g)(a^p+b^p-y)\big]$$
$$=h(t)\overline{(f\circ g)}(x)+h(1-t)\overline{(f\circ g)}(y).$$

Thus $f\circ g$ is symmetrized h-convex on $g^{-1}([a,b])$.

Conversely, if $f\circ g$ is symmetrized h-convex on $g^{-1}([a,b])$ then it is easily seen that f is symmetrized (p,h)-convex by a similar procedure. The details are omitted. □

Theorem 3.5. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized (p,h)-convex and integrable on $[a,b]$ with h is integrable on $[0,1]$. Then we have the Hermite-Hadamard-type inequality*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\le\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx$$
$$\le\big[f(a)+f(b)\big]\int_0^1 h(x)dx. \quad (3.1)$$

Proof. Since $f:[a,b]\to\mathbb{R}$ is symmetrized (p,h)-convex on $[a,b]$, then by writing the Hermite-Hadamard-type inequality (1.1) for the function $P_{(f;p)}$ we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}P_{(f;p)}\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\le\frac{p}{b^p-a^p}\int_a^b\frac{P_{(f;p)}(x)}{x^{1-p}}dx$$
$$\le\big[P_{(f;p)}(a)+P_{(f;p)}(b)\big]\int_0^1 h(x)dx. \quad (3.2)$$

It is easily seen that

$$P_{(f;p)}\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)=f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right),$$
$$P_{(f;p)}(a)+P_{(f;p)}(b)=f(a)+f(b),$$

and

$$\frac{p}{b^p-a^p}\int_a^b\frac{P_{(f;p)}(x)}{x^{1-p}}dx=\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx.$$

Then by (3.2), required inequality are got.

Remark 3.6. In Theorem 3.5, if it is chosen $p=1$ then the inequality (3.1) reduce to the inequality (1.2). If it is chosen $p=-1$ then the inequality (3.1) reduce to the inequality (1.3).

Remark 3.7. By helping Proposition 3.4, the proof of Theorem 3.5 can also be given as follows:

Since $f:[a,b]\to\mathbb{R}$ is symmetrized (p,h)-convex on the interval $[a,b]$, $f\circ g$ is symmetrized h-convex on the interval $g^{-1}([a,b])=[a^p,b^p]$ (or $[b^p,a^p]$) with $g(x)=x^{1/p}, x>0, p\neq 0$. So by inequality (1.2), it is obtained that

$$\frac{1}{2h\left(\frac{1}{2}\right)}(f\circ g)\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b (f\circ g)(x)dx$$

$$\le [(f\circ g)(a)+(f\circ g)(b)]\int_0^1 h(x)dt,$$

i.e.

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\le \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx$$

$$\le \left[f(a)+f(b)\right]\int_0^1 h(x)dx.$$

Theorem 3.8. *If $f:[a,b]\to\mathbb{R}$ is symmetrized (p,h)-convex on $[a,b]$, then for any $x\in[a,b]$, we have the following inequality*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\le P_{(f;p)}(x)$$

$$\le \left[h\left(\frac{x^p-a^p}{b^p-a^p}\right)+h\left(\frac{b^p-x^p}{b^p-a^p}\right)\right]\frac{f(a)+f(b)}{2}. \quad (3.3)$$

Proof. Since $P_{(f;p)}$ is (p,h)-convex on $[a,b]$ then for any $x\in[a,b]$ we have

$$P_{(f;p)}\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)$$

$$\le h\left(\frac{1}{2}\right)\left[P_{(f;p)}(x)+P_{(f;p)}\left([a^p+b^p-x^p]^{1/p}\right)\right]$$

and since

$$P_{(f;p)}(x)+P_{(f;p)}\left([a^p+b^p-x^p]^{1/p}\right)$$

$$= f(x)+f\left([a^p+b^p-x^p]^{1/p}\right)=2P_{(f;p)}(x)$$

while

$$P_{(f;p)}\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)=f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right),$$

we get the first inequality in (3.3).

Also, by the (p,h)-convexity of $P_{(f;p)}$ we have for any $x\in[a,b]$ that

$$P_{(f;p)}(x)=P_{(f;p)}\left(\left[b^p\frac{x^p-a^p}{b^p-a^p}+a^p\frac{b^p-x^p}{b^p-a^p}\right]^{1/p}\right)$$

$$\le h\left(\frac{x^p-a^p}{b^p-a^p}\right)P_{(f;p)}(b)+h\left(\frac{b^p-x^p}{b^p-a^p}\right)P_{(f;p)}(a)$$

$$=\left[h\left(\frac{x^p-a^p}{b^p-a^p}\right)+h\left(\frac{b^p-x^p}{b^p-a^p}\right)\right]\frac{f(a)+f(b)}{2}$$

which gives the second part of (3.3).

Remark 3.9. In Theorem 3.8, if we choose $p=1$, then the inequality (3.3) reduce to the inequality (3.18) in Theorem 7 in (Dragomir, 2016). If we choose $p=-1$ then the inequality (3.3) reduce to the inequality (4.3) in Theorem 6 in (Wu & et al., 2017) and if it is chosen $h(\alpha)=\alpha, \alpha\in(0,1)$, then the inequality (3.3) reduce to the inequality (18) in Theorem 2.2 in (İşcan, 2020).

Remark 3.10. By helping Proposition 3.4 and Theorem 7 in (Dragomir, 2016), the proof of Theorem 3.8 can also be given. The details are obmitted.

Corollary 3.11. *Assume that the function $f:[a,b]\to\mathbb{R}$ is symmetrized (p,h)-convex and integrable on $[a,b]$ with h is integrable on $[0,1]$. If $w:[a,b]\to[0,\infty)$ is integrable on $[a,b]$ then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx$$

$$\le \int_a^b \frac{w(x)P_{(f;p)}(x)}{x^{1-p}}dx$$

$$\le \frac{f(a)+f(b)}{2}\int_a^b\left[h\left(\frac{x^p-a^p}{b^p-a^p}\right)+h\left(\frac{b^p-x^p}{b^p-a^p}\right)\right]\frac{w(x)}{x^{1-p}}dx. \quad (3.4)$$

Moreover, if w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ on $[a,b]$, i.e. $w(x)=w\left([a^p+b^p-x^p]^{1/p}\right)$, for all $x\in[a,b]$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx$$

$$\le \int_a^b \frac{w(x)f(x)}{x^{1-p}}dx$$

$$\le [f(a)+f(b)]\int_a^b h\left(\frac{x^p-a^p}{b^p-a^p}\right)\frac{w(x)}{x^{1-p}}dx. \quad (3.5)$$

Proof. The inequality (3.4) follow by (3.3) multiplying by $w(x)/x^{1-p} \geq 0$ and integrating over x on $[a,b]$.

By changing variables and using the fact that w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, we have

$$\int_a^b \frac{w(x) f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)}{x^{1-p}}dx$$

$$=\int_a^b \frac{w\left([a^p+b^p-x^p]^{1/p}\right) f(x)}{x^{1-p}}dx=\int_a^b \frac{w(x)f(x)}{x^{1-p}}dx$$

and

$$\int_a^b h\left(\frac{b^p-x^p}{b^p-a^p}\right)\frac{w(x)}{x^{1-p}}dx$$

$$=\int_a^b h\left(\frac{x^p-a^p}{b^p-a^p}\right)\frac{w\left(\left[a^p+b^p-x^p\right]^{1/p}\right)}{x^{1-p}}dx.$$

$$=\int_a^b h\left(\frac{x^p-a^p}{b^p-a^p}\right)\frac{w(x)}{x^{1-p}}dx.$$

Thus

$$\int_a^b \frac{w(x)P_{(f;p)}(x)}{x^{1-p}}dx$$

$$=\frac{1}{2}\left[\int_a^b \frac{w(x)f(x)}{x^{1-p}}dx+\int_a^b \frac{w(x)f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)}{x^{1-p}}dx\right]$$

$$=\int_a^b \frac{w(x)f(x)}{x^{1-p}}dx \qquad (3.6)$$

and

$$\frac{f(a)+f(b)}{2}\int_a^b\left[h\left(\frac{x^p-a^p}{b^p-a^p}\right)+h\left(\frac{b^p-x^p}{b^p-a^p}\right)\right]\frac{w(x)}{x^{1-p}}dx$$

$$=[f(a)+f(b)]\int_a^b h\left(\frac{x^p-a^p}{b^p-a^p}\right)\frac{w(x)}{x^{1-p}}dx. \quad (3.7)$$

By (3.6), (3.7), and (3.4), the inequality (3.5) are got.

Remark 3.12. If it is chosen as $p=1$, then the results in Corollary 3.11 reduce to the results in Corollary 2 in (Dragomir, 2016) and if it is chosen $p=-1$ then the inequality (3.4) reduce to the inequality (4.4) in Corollary 1 in (Wu & et al., 2017).

Remark 3.13. By helping Proposition 3.4 and Corollary 2 in (Dragomir, 2016), the proof of Theorem 3.11 can also be given. The details are omitted.

Theorem 3.14. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized (p,h)-convex and integrable on $[a,b]$ with h is integrable on $[0,1]$. Then for any $x,y\in[a,b]$, we have the following inequality*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)+f\left(\left[a^p+b^p-\frac{x^p+y^p}{2}\right]^{1/p}\right)\right]$$

$$\leq\frac{p}{y^p-x^p}\left[\int_x^y\frac{f(t)}{t^{1-p}}dt+\int_{[a^p+b^p-y^p]^{1/p}}^{[a^p+b^p-x^p]^{1/p}}\frac{f(t)}{t^{1-p}}dt\right]$$

$$\leq\left[f(x)+f(y)+f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)+\right.$$

$$\left.f\left(\left[a^p+b^p-y^p\right]^{1/p}\right)\right]\int_0^1 h(t)dt. \qquad (3.8)$$

Proof. Since $P_{(f;p)}$ is (p,h)-convex on $[a,b]$, then $P_{(f;p)}$ is also (p,h)-convex on any sub-interval $[x,y]$ (or $[y,x]$), where $x,y\in[a,b]$.

By writing Hermite-Hadamard (1.1) for $P_{(f;p)}$ on $[x,y]$ we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}P_{(f;p)}\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)\leq\frac{p}{y^p-x^p}\int_x^y\frac{P_{(f;p)}(t)}{t^{1-p}}dt$$

$$\leq\left[P_{(f;p)}(a)+P_{(f;p)}(b)\right]\int_0^1 h(t)dt. \qquad (3.9)$$

By the definition of $P_{(f;p)}$, it is easily seen that

$$P_{(f;p)}\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)$$

$$=\frac{1}{2}\left[f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)+f\left(\left[a^p+b^p-\frac{x^p+y^p}{2}\right]^{1/p}\right)\right],$$

$$\int_x^y\frac{P_{(f;p)}(t)}{t^{1-p}}dt$$

$$=\frac{1}{2}\int_x^y\frac{f(t)+f\left(\left[a^p+b^p-t^p\right]^{1/p}\right)}{t^{1-p}}dt$$

$$=\frac{1}{2}\int_x^y\frac{f(t)}{t^{1-p}}dt+\frac{1}{2}\int_x^y\frac{f\left(\left[a^p+b^p-t^p\right]^{1/p}\right)}{t^{1-p}}dt$$

$$=\frac{1}{2}\int_x^y\frac{f(t)}{t^{1-p}}dt+\frac{1}{2}\int_{\left[a^p+b^p-y^p\right]^{1/p}}^{\left[a^p+b^p-x^p\right]^{1/p}}\frac{f(t)}{t^{1-p}}dt$$

and

$$P_{(f;p)}(x)+P_{(f;p)}(y)$$

$$=\frac{1}{2}\left[f(x)+f(y)+f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)\right.$$

$$\left.+f\left(\left[a^p+b^p-y^p\right]^{1/p}\right)\right].$$

By (3.9), we deduce the desired result (3.8).

Remark 3.15. If we choose $p=1$, then (3.8) reduces inequality (3.14) in Theorem 6 in (Dragomir, 2016) and if it is chosen $p=-1$ then the inequality (3.8) reduce to the inequality (4.1) in Theorem 5 in (Wu & et al., 2017).

Remark 3.16. By helping Proposition 3.4 and Theorem 6 in (Dragomir, 2016), the proof of Theorem 3.14 can also be given. The details are omitted.

4. CONCLUSION

The paper established some Hermite-Hadamard-type inequalities for symmetrized *(p,h)*-convex functions which is a generalization of symmetrized harmonic *h*-convex functions, symmetrized *p*-convex functions, and symmetrized *h*-convex functions. We can see that those results are extensions of the results in the previous research.

TÍNH *(p,h)*-LỒI ĐỐI XỨNG HÓA VÀ MỘT SỐ BẤT ĐẲNG THỨC KIỂU HERMITE-HADAMARD

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TÓM TẮT

Bài báo này giới thiệu hàm *(p,h)*-lồi đối xứng hóa và thiết lập một số bất đẳng thức kiểu Hermite-Hadamard cho lớp hàm mới.

Từ khóa: *(p,h)-lồi đối xứng hóa, (p,h)-lồi, p-lồi, h-lồi, lồi điều hòa, bất đẳng thức Hermite-Hadamard.*

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