RESEARCH ON NONLINEAR FREE VIBRATIONS USING HOMOTOPY ANALYSIS METHOD

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ABSTRACT

This paper applies the homotopy analysis method to study nonlinear free vibrations. The paper has used the homotopy analysis method to find analytical solutions of a nonlinear two-degree-of-freedom system. Analytical results have found the natural oscillation frequencies, the displacement formula of the system are approximately calculated at the 10th order. Research to find analytical solutions only receives positive results when selecting the auxiliary parameter $\hbar$ is the deciding factor in the values of the solution. Compared with the results calculated by numerical methods, using calculations by homotopy analysis method is an advantage to find solutions to the nonlinear vibration problem. From calculating results, in case of auxiliary parameter $\hbar = -0.1$, those results are matched very well when analyzed by the numerical methods. In additionally, to evaluate the reliability of the homotopy analysis method, this paper has determined that when the time variable is in the range t = $[0 \rightarrow 5.4\pi]$, the displacemental results are compared by calculated by the homotopy method with the numerical method would be confidence interval.

NGHIÊN CỨU DAO ĐỘNG TỰ DO PHI TUYẾN SỬ DỤNG PHƯƠNG PHÁP PHÂN TÍCH ĐỒNG LUÂN

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TỪ KHÓA

Phân tích đồng luân
Phi tuyến
Xấp xỉ
Tần số tự nhiên
Chuyển vị ngang

TÓM TẮT

Bài báo này áp dụng phương pháp phân tích đồng luân để nghiên cứu dao động tự do phi tuyến. Bài báo dùng phương pháp phân tích đồng luân để tìm nghiệm giải tích của hệ hai bậc tự do phi tuyến. Kết quả giải tích đã tìm được tần số dao động riêng, công thức chuyển vị của hệ được xấp xỉ tính toán ở bậc 10. Việc nghiên cứu tìm nghiệm giải tích chỉ nhận được kết quả khả quan khi lựa chọn tham số phụ $\hbar$ là yếu tố quuyết định đến giá trị của nghiệm. So sánh với kết quả tính bằng phương pháp số, sử dụng tính toán bằng phương pháp phân tích đồng luân là một lợi thế để tìm nghiệm của bài toán dao động phi tuyến. Từ kết quả tính toán của bài báo, khi tham số phụ $\hbar = -0.1$ nhận được kết quả trùng khớp với kết quả giải bằng phương pháp số. Ngoài ra, để đánh giá mức độ tin cậy trong phương pháp phân tích đồng luân, bài báo đã xác định được khi biến thời gian trong khoảng $t = [0 \rightarrow 5.4\pi]$, kết quả so sánh về chuyển vị tính bằng phương pháp phân tích đồng luân với phương pháp số là đáng tin cậy.

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1. Introduction

Nonlinear vibration problems cannot always be solved with analytical solutions, especially problems with high nonlinearity. Research on solving nonlinear vibration problems using analytical methods has been discussed by many authors [1] - [5]. The authors [1] – [3], [5] used the perturbation method; the asymptotic method was used in [4]; the homotopy analysis method (HAM) was used in [6] – [10] and the homotopy perturbation method was used in [9] – [13]. Solving the nonlinear vibration problem by analytical methods with perturbation direction and asymptotic methods requires that a small parameter exists in the equation, from which the series can be expanded according to that small parameter [1] - [5]. However, depending on small parameters sometimes makes it difficult to find solutions to highly nonlinear differential equations. Therefore, choosing an analytical method to solve highly nonlinear problems requires something worth paying attention to when choosing a solution method.

With the introduction of the homotopy analysis method [6] - [10], it is one of the methods for finding analytical solutions of the nonlinear vibration problem with certain advantages. The outstanding advantage is that there is no need for small parameters in the equation, so it can be applied to nonlinear problems in the most natural way. The characteristic of the homotopy analysis method is to build a class of problems with the participation of the embedded parameter q = [0,1], from which to build functions and variables in the form of power series expansions of the embedded parameter q.

Applying the homotopy analysis method (HAM) to solving nonlinear problems sometimes requires knowing how to flexibly select input factors appropriate to the problem to be solved, so to get positive results requires a considerable experience and skills. The input requirements are formula representing the rule of solution expression, choozing the initial approximation, building the linear auxiliary operator, the auxiliary parameter $\hbar$, and the auxiliary function. The received results are also an expression containing the auxiliary parameter $\hbar$, so to find a suitable solution for the original nonlinear problem, it is necessary to see at what value of the auxiliary parameter the received series will converge to get the required result.

In this paper, the homotopy analysis method has been applied to analyze the problem of nonlinear free vibration for a two-degree-of-freedom system, thereby determining the solution expression as two horizontal displacements $u_1(t)$ and $u_2(t)$, and at the same time received the natural oscillation frequency ω. The results show that corresponding to the value $\hbar = -0.1$, results consistent with the original nonlinear problem are obtained. When the value of the auxiliary parameter $\hbar = [-0.15 \to 0.04]$, the series converges to the solution of the problem corresponding to the 10th order solution.

By applying the homotopy method to find the solution of the nonlinear vibration problem, this paper has obtained the solution expression of the problem by series expansion at the 10th order. The paper have already compared the results of applying by HAM with the numeric method results. The results were evaluated with the confidence interval of the homotopy analysis method.

The contents of this paper consist of sections: 1. Introduction; 2. The method of vibration analysis; 3. Results and discussion; 4. Conclusions.

2. The method of vibration analysis

In the article [14], the author models a 2-storey house structure described by a two-degree-of-freedom nonlinear vibration equation, with the unknowns being two horizontal displacements $u_1(t), u_2(t)$. To solve the problem of undamped free oscillation, we get the following equation:

$$\ddot{u}_1 + \chi_{11}u_1 + 4\gamma_1 u_1^3 + 2\gamma_3 u_1 u_2^2 = 0$$
$$\ddot{u}_2 + \chi_{22}u_2 + 4\gamma_5 u_2^3 = 0 \quad (1)$$

with the boundary conditions:

$$u_1(0) = a_0;\ u_1{}'(0) = 0;\ u_2(0) = -b_0; u_2{}'(0) = 0 \quad (2)$$

In the equations (1) the parameters $\chi_{11}, \chi_{22}, \gamma_1, \gamma_3, \gamma_5$ are given values. In this article, we will not survey the change of these parameters to the results of $u_1(t), u_2(t)$ because we will receive the solution expression containing those parameters. Our task now is to solve the nonlinear differential equation (1) by using analytical methods, that is, to find two horizontal displacements $u_1(t), u_2(t)$.

Rewrite equations (1) as following:

$$\begin{aligned} &\frac{d^2u_1}{dt^2} + \chi_{11}u_1 + 4\gamma_1 u_1^3 + 2\gamma_3 u_1 u_2^2 = 0; \\ &\frac{d^2u_2}{dt^2} + \chi_{22}u_2 + 4\gamma_5 u_2^3 = 0 \end{aligned} \tag{3}$$

Use transformations:

$$\tau = \omega t, u_1(\tau) = u(t), u_2(t) = \delta v(\tau),$$
$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \omega\frac{d}{d\tau}, \frac{d^2}{dt^2} = \omega^2\frac{d^2}{d\tau^2}, \tag{4}$$

Equations (3) becomes:

$$\begin{cases} \omega^2 \dfrac{d^2u}{d\tau^2} + \chi_{11}u + 4\gamma_1 u^3 + 2\gamma_3\delta^2 u v^2 = 0 \\ \omega^2 \dfrac{d^2v}{d\tau^2} + \chi_{22}u + 4\gamma_5\delta^2 v^3 = 0 \end{cases} \tag{5}$$

The boundary conditions in equations (2) have been chosen $a_0=1$, $b_0=1$, then equations (2) become:

$$u(0) = 1, u'(0) = 0,$$
$$v(0) = -1, v'(0) = 0, \tag{6}$$

Thus, the original nonlinear equations (1) with two unknowns - two displacements $u_1(t)$ and $u_2(t)$ have been replaced by solving equations (5) with two new displacements $u(\tau), v(\tau)$ and two unknown parameters ω, δ. The following section uses the homotopy analysis method (HAM) to find approximate analytical solutions for equation (5) with boundary condition (6).

To apply the HAM to solve the system of nonlinear equations (5) with boundary conditions (6), we choose the solution expression form as:

$$u(\tau) = u_0(\tau) + \sum_{n=1}^{+\infty}[a_n \cos(2n+1)\tau], v(\tau) = v_0(\tau) + \sum_{n=1}^{+\infty}[b_n \cos(2n+1)\tau] \tag{7}$$

in which the initial approximation solution is chosen to satisfy the boundary condition (6):

$$u_0(\tau) = \cos(\tau), v_0(\tau) = -\cos(\tau), \tag{8}$$

According to the HAM, choose the auxiliary linear operator as:

$$\Im_u[\Phi(\tau;q)] = \left[\frac{\partial^2\Phi(\tau;q)}{\partial\tau^2} + \Phi(\tau;q)\right], \Im_v[\Theta(\tau;q)] = \left[\frac{\partial^2\Theta(\tau;q)}{\partial\tau^2} + \Theta(\tau;q)\right] \tag{9}$$

with the property:

$$\Im_u[C_1 \sin\tau + C_2 \cos\tau] = 0 \tag{10}$$

In order to solve equations (5) by the HAM, we define the nonlinear operator:

$$\aleph_u[\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)] =$$
$$= \Omega^2(q)\frac{\partial^2\Phi(\tau;q)}{\partial\tau^2} + \chi_{11}\Phi(\tau;q) + 4\gamma_1(\Phi(\tau;q))^3 + 2\gamma_3(\Delta(q))^2\Phi(\tau;q)(\Theta(\tau;q))^2 \tag{11}$$

$$\aleph_v[\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)] =$$
$$= \Omega^2(q)\frac{\partial^2\Theta(\tau;q)}{\partial\tau^2} + \chi_{22}\Delta(q)\Theta(\tau;q) + 4\gamma_5(\Delta(q))^2(\Theta(\tau;q))^3 \tag{12}$$

where $\Phi(\tau;q), \Theta(\tau;q)$ is a function of τ and q; $\Omega(q), \Delta(q)$ is a function of embedding parameter q $(0\le q\le 1)$. Let $\hbar_1, \hbar_2$ are auxiliary parameters not equal to zero, then to construct the zero-order deformation equations as following:

$$\begin{cases} (1-q)\Im_u[\Phi(\tau;q) - u_0(\tau)] = q\hbar_1\aleph_u[\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)] \\ (1-q)\Im_v[\Theta(\tau;q) - v_0(\tau)] = q\hbar_2\aleph_v[\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)] \end{cases} \tag{13}$$

will match with initial boundary conditions:

$$\Phi(0;q) = a_0, \frac{\partial \Phi(\tau;q)}{\partial \tau}\Big|_{q=0} = 0$$

$$\Theta(0;q) = b_0, \frac{\partial \Phi(\tau;q)}{\partial \tau}\Big|_{q=0} = 0 \quad (14)$$

When q=0, from the zero-order deformation equations we get:

$$\Phi(\tau;0) = u_0(\tau), \Theta(\tau;0) = v_0(\tau), \quad (15)$$

When q=1, from the zero-order deformation equations we get:

$$\Phi(\tau;1) = u(\tau), \Theta(\tau;1) = v(\tau), \Delta(1) = \delta, \Omega(1) = \omega \quad (16)$$

By Taylor's theorem, we expand $\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)$ in a power series of the embedding parameter *q* as following:

$$\Phi(\tau;q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) q^m, \Theta(\tau;q) = v_0(\tau) + \sum_{m=1}^{+\infty} v_m(\tau) q^m \quad (17)$$

$$\Omega(q) = \omega_0 + \sum_{m=1}^{+\infty} \omega_m(\tau) q^m, \Delta(q) = \delta_0 + \sum_{m=1}^{+\infty} \delta_m(\tau) q^m \quad (18)$$

when q=1, those series converge and then get:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), v(\tau) = v_0(\tau) + \sum_{m=1}^{+\infty} v_m(\tau) \quad (19)$$

$$\omega = \omega_0 + \sum_{m=1}^{+\infty} \omega_m(\tau), \delta = \delta_0 + \sum_{m=1}^{+\infty} \delta_m(\tau) \quad (20)$$

In the equations (17), (18), denote the following quantities:

$$u_m(\tau) = \frac{1}{m!} \frac{\partial^m \Phi(\tau;q)}{\partial q^m}\Big|_{q=0}, v_m = \frac{1}{m!} \frac{\partial^m \Theta(q)}{\partial q^m}\Big|_{q=0} \quad (21)$$

$$\delta_m = \frac{1}{m!} \frac{\partial^m \Delta(q)}{\partial q^m}\Big|_{q=0}, \omega_m = \frac{1}{m!} \frac{\partial^m \Omega(q)}{\partial q^m}\Big|_{q=0}, \quad (22)$$

From the zero-order deformation equations (13), the m^{th}-order derivative with respect to embedding parameter *q* we will get the m^{th}-order deformation equation:

$$\Im_u[u_m(\tau) - \lambda_m u_{m-1}(\tau)] = \hbar_1 R_m^u(\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1})$$

$$\Im_v[v_m(\tau) - \lambda_m v_{m-1}(\tau)] = \hbar_2 R_m^v(\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) \quad (23)$$

subject to the initial conditions:

$$u_m(0) = 0, u'_m(0) = 0, v_m(0) = 0, v'_m(0) = 0, \quad (24)$$

where $\lambda_m = 0 (m <= 1), \lambda_m = 1 (m > 1)$, $\hbar_1, \hbar_2$- auxiliary parameters. When implemented by the HAM, $\hbar_1, \hbar_2$ are meaningful to determine the convergence region of the series (17), (18).

The right-hand side of equation (23) is written as the *m-1* order derivative with respect to *q*:

$$R_m^u(\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) = \frac{1}{(m-1)!} \frac{d^{m-1} \aleph_u[\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)]}{dq^{m-1}}\Big|_{q=0}$$

$$R_m^v(\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) = \frac{1}{(m-1)!} \frac{d^{m-1} \aleph_v[\Phi(\tau;q), \Theta(\tau;q), \Omega(q), \Delta(q)]}{dq^{m-1}}\Big|_{q=0} \quad (25)$$

in equation (25) use symbols:

$$\vec{u}_m = (\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_m), \vec{v}_m = (\vec{v}_m, \vec{v}_m, \ldots, \vec{v}_m), \vec{\omega}_m = (\vec{\omega}_0, \vec{\omega}_1, \ldots, \vec{\omega}_m), \vec{\delta}_m = (\vec{\delta}_0, \vec{\delta}_1, \ldots, \vec{\delta}_m) \quad (26)$$

From the assumption that the solution expression is written as equation (7), then the right side of equation (25) is written as:

$$R_m^u(\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) = \sum_{n=0}^{\phi(m)} a_{m,n}(\vec{\omega}_{m-1}, \vec{\delta}_{m-1}) \cos[(2n+1)\tau]$$

$$R_m^v(\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) = \sum_{n=0}^{\varphi(m)} b_{m,n}(\vec{\omega}_{m-1}, \vec{\delta}_{m-1}) \cos[(2n+1)\tau] \quad (27)$$

From equations (23), the solution expression is written by:

$$u_m(\tau) = \lambda_m u_{m-1}(\tau) + \hbar_1 \sum_{n=2}^{\phi(m)} \frac{a_{m,n}(\vec{\omega}_{m-1}, \vec{\delta}_{m-1})}{(1-n^2)} \cos[(2n+1)\tau]$$

$$v_m(\tau) = \lambda_m v_{m-1}(\tau) + \hbar_2 \sum_{n=2}^{\phi(m)} \frac{b_{m,n}(\vec{\omega}_{m-1}, \vec{\delta}_{m-1})}{(1-n^2)} \cos[(2n+1)\tau] \quad (28)$$

The mth-order approximation solution M, will get as:

$$u(\tau) \approx \sum_{m=0}^{M} u_m(\tau)\,, v(\tau) \approx \sum_{m=0}^{M} v_m(\tau) \quad (29)$$

$$\omega \approx \sum_{m=0}^{M} \omega_m(\tau)\,, \delta \approx \sum_{m=0}^{M} \delta_m(\tau) \quad (30)$$

Finally, we will get the displacements of the original equation (1):

$$u(\tau) = u_1(\omega t) = \sum_{m=0}^{M} u_m(\omega t)\,, u_2(\tau) = \delta v(\omega t) = \sum_{m=0}^{M} \delta_m \sum_{m=0}^{M} u_m(\omega t) \quad (31)$$

3. Results and discussion

With the initial approximation solution $u_0(\tau) = \cos(\tau), v_0(\tau) = -\cos(\tau)$, parametes ω_0, δ_0 can be calculated by the following equations as:

$$\omega_0 = \pm\sqrt{\frac{2(\chi_{11}-\chi_{22}+3\gamma_1)}{3(2\gamma_5-\gamma_3)}}, \delta_0 = \pm\sqrt{\frac{(2\chi_{11}\gamma_5-\chi_{22}\gamma_3+6\gamma_1\gamma_5)}{3(2\gamma_5-\gamma_3)}} \quad (32)$$

and also get the first-order approximation solution as:

$$u_1(\tau) = \frac{\hbar_1}{16}(2\gamma_1 + \gamma_3\delta_0^2)(\cos\tau - \cos 3\tau), v_1(\tau) = -\frac{\hbar_2}{8}\gamma_5\delta_0^2(\cos\tau - \cos 3\tau), \quad (33)$$

For higher order approximations we also get the same instinctive formula as above. But, because expressing them is very long, to make it easier to see the higher order approximations, here assigned the parameters in the original nonlinear equation systems (1) as values of $\chi_{11} = 5.013226625, \chi_{22} = 2.466371557, \gamma_1 = 1, \gamma_3 = 1, \gamma_5 = 1, \hbar_1 = -0.1, \hbar_2 = -0.1,$ thereby obtaining approximate values for the first three orders as shown in Table 1.

Table 1. *Calculated results by HAM with the first three orders*

Order No	ω	δ	$u(\tau)$	$v(\tau)$
0			$cos(\tau)$	$-cos(\tau)$
1	3.682401	-1.922993	$-0.03561189611cos(\tau)$ $+ 0.03561189611cos(3\tau)$	$0.04622379222cos(\tau)$ $- 0.04622379222cos(3\tau)$
2	-0.03434	-0.022523	$0.01021447222cos(\tau)$ $- 0.01164618673cos(3\tau)$ $+ 0.001431714505cos(5\tau)$	$-0.01323657449cos(\tau)$ $+ 0.01537321346cos(3\tau)$ $- 0.002136638969cos(5\tau)$
3	0.009879	0.0074637	$-0.00347715794cos(\tau)$ $+ 0.00436112291cos(3\tau)$ $- 0.000943726937cos(5\tau)$ $+ 0.00005976196621cos(7\tau)$	$0.00437453843cos(\tau)$ $- 0.00569699132cos(3\tau)$ $+ 0.001421216446cos(5\tau)$ $- 0.00009876355cos(7\tau)$
Σ1+2 + 3order	3.657940	-1.938053	$0.9711254179cos(\tau)$ $+ 0.02832683229cos(3\tau)$ $+ 0.000487987568cos(5\tau)$ $+ 0.00005976196621cos(7\tau)$	$-0.9626382438cos(\tau)$ $- 0.03654757008cos(3\tau)$ $- 0.000715422523cos(5\tau)$ $- 0.00009876355cos(7\tau)$

Table 1 shows that the results calculated by HAM are obtained as expression the values of the variables according to the initial parameters and boundary conditions. Most of the methods for solving nonlinear equations use other methods that cannot give expressions, but the HAM method has shown the results of expressions, depending on the level of accuracy of the problem we are trying to solve to what level. In this paper, we have calculated the results up to the 10th order, and the results are expressed by solution expressions as in table 1. The obtained results, in addition to depending on boundary conditions and model parameters $\chi_{11}, \chi_{22}, \gamma_1, \gamma_3, \gamma_5$, also depend on the two auxiliary parameters $\hbar_1, \hbar_2$ shown in Table 2.

Figures 1 to 5 are the results on the graph calculated by the HAM and by numerical methods. Figure 1 shows the numerical results for the original equation (1) with boundary conditions *$u_1(0)=1$, $u_2(0)= 1.939975786$*, showing the horizontal axis as $t = 0 \rightarrow 2\pi$, the vertical axis are two displacements: *CV1num* is the displacement *$u_1(t)$* and *CV2num* is the displacement *$u_2(t)$*.

Figure 2 is the displacement results by HAM of equation (5) with boundary conditions $u(0) = 1, v(0) = -1$, displaying the horizontal axis as $t = 0 \rightarrow 2\pi$, vertical axis are two displacements: *CV1* is the displacement $u_1(\tau) = u(\omega t)$ and *CV2* is the displacement $u_2(\tau) = \delta v(\omega t)$, with the values δ, ω and $u(\tau), v(\tau)$ determined at the 10th iteration and assigned the value $\hbar_1 = -0.1, \hbar_2 = -0.1$. This result is equivalent to solving the original equation (1) by numeric methods as shown in Figure 1. This is a harmonic oscillation with period T = 2π/ω, with $\omega = \sum_{i=0}^{9} \omega_i = 3.6555 rad/s$ (approximate solution of the series (31) to the 10th order).

Table 2. *Calculated results by HAM with the three orders without assigning a value of* h_1, h_2

No	Calculated results completed at the first three orders for the following
	$u(\tau) = u_0(\tau) + u_1(\tau) + u_2(\tau) + u_3(\tau)$
1	$cos(\tau) + 1.068356883cos(\tau)\hbar_1 - 1.068356883\hbar_1 cos(3\tau) + 2*(4.938743330\hbar_1^2 - 0.3561064946\hbar_1\hbar_2)cos(\tau) + 2\hbar_1*\left(-\left((40.08554783\hbar_1 - 2.279081565\hbar_2)cos(3\tau)\right)/8 - cos(5\tau)*(-1.726803639\hbar_1 - 1.709311174\hbar_2)/24\right) + (68.71576922\hbar_1^3 + (-4.464957666\hbar_2 + 4.938743326)\hbar_1^2 + (-4.732813039\hbar_2^2 - 0.3561064950\hbar_2)\hbar_1)cos(\tau) + \hbar_1(-cos(3\tau)(-31.08011189\hbar_2^2 - 2.279081568\hbar_2 + 565.6241001\hbar_1^2 + 40.08554783\hbar_1 - 28.42084837\hbar_1\hbar_2)/8 - cos(5\tau)(-1.726803638\hbar_1 - 1.709311174\hbar_2 - 20.93975857\hbar_2^2 - 48.05851198\hbar_1^2 - 22.37347221\hbar_1\hbar_2)/24 - cos(7\tau)(1.185162669\hbar_2^2 + 0.7293449527\hbar_1^2 + 0.9540667568\hbar_1\hbar_2)/48)$
	$v(\tau) = v_0(\tau) + v_1(\tau) + v_2(\tau) + v_3(\tau)$
2	$-cos(\tau) - 1.386713767\hbar_2 cos(\tau) + 1.386713767\hbar_2 cos(3\tau) + 2(-5.911877492\hbar_2^2 - 0.03415917762\hbar_1\hbar_2)cos(\tau) + 2\hbar_2(-((-49.00433112\hbar_2 - 0.2732734210\hbar_1)cos(3\tau))/8 - 0.2136638969\hbar_2 cos(5\tau)) - 0.5235500099\hbar_2(144.6924053\hbar_2^2 + (1.516988494\hbar_1 + 11.29190599)\hbar_2 + (\hbar_1 + 0.06524529983)\hbar_1)cos(\tau) + \hbar_2(-cos(3\tau)(-4.188400080\hbar_1^2 - 0.2732734190\hbar_1 - 49.00433114\hbar_2 - 650.5428940\hbar_2^2 - 6.606389408\hbar_1\hbar_2)/8 - cos(5\tau)(135.9099611\hbar_2^2 + 5.127933522\hbar_2 + 0.7579040302\hbar_1\hbar_2)/24 + 0.09876355571cos(7\tau)\hbar_2^2)$
	$\omega = \omega_0 + \omega_1 + \omega_2$
3	$1.704432363\hbar_1^2 + (0.2226319517 + 0.04259761932\hbar_2)\hbar_1 + 3.682401620 + 0.4641837982\hbar_2 + 2.674983770\hbar_2^2$
	$\delta = \delta_0 + \delta_1 + \delta_2$
4	$-1.922993338 + 0.5925869646\hbar_2 - 0.1421083551\hbar_1 - 1.087718637\hbar_1^2 + 4.063416522\hbar_2^2 + 0.02307102239\hbar_1\hbar_2$

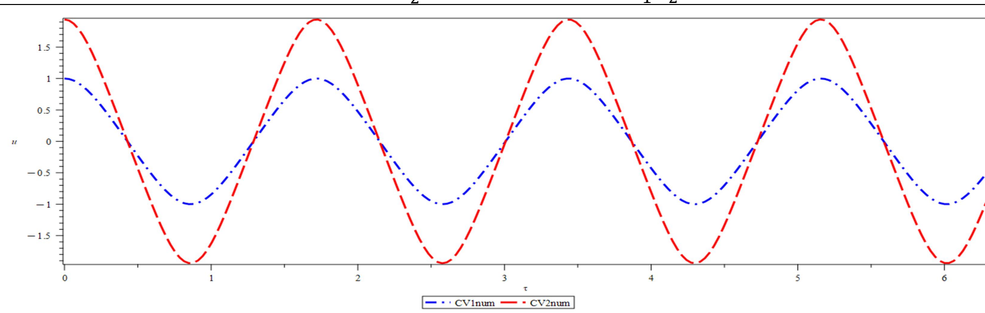

Figure 1. *Calculated by numerical methods in the range* $t = 0 \to 2\pi$

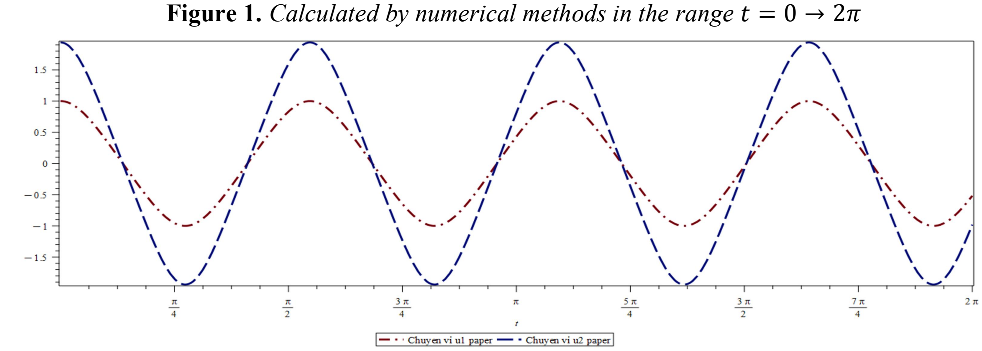

Figure 2. *Calculated by HAM with* $\hbar_1 = -0.1, \hbar_2 = -0.1$

Figure 3 shows two images in the Figures 1 and 2 drawn on the same coordinate axis for ease of comparison when calculated by the two methods, thereby showing that using the HAM method and the numerical method, the displacement value corresponds to $t = 0 \rightarrow 4\pi$ coincide closely.

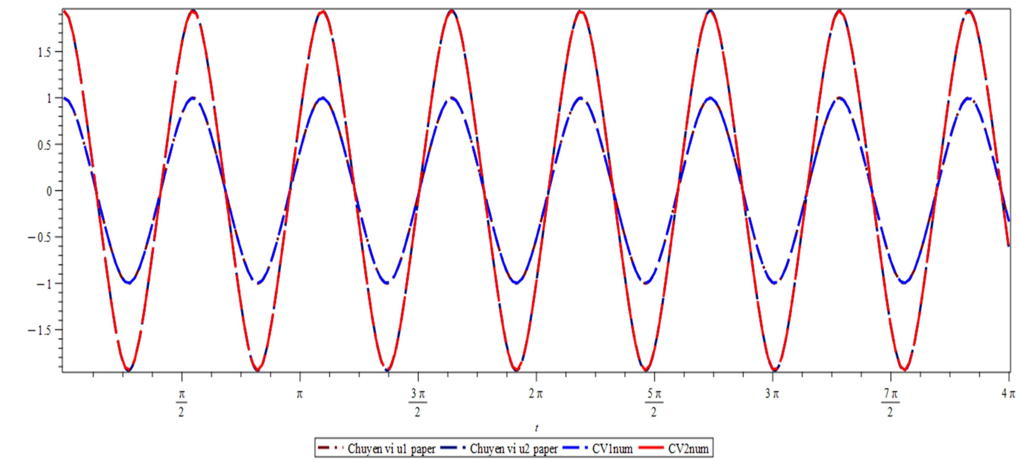

Figure 3. *Two images in Figure 1 and 2 combined into the same coordinates*

Figure 4 shows the displacement results by using two methods HAM and the numerical method. The range of $t = 4\pi \rightarrow 8\pi$ was considered, we have found that the displacement u_1 is not accurate from the range $t > 5.4\pi$, but u_2 still matches closly ($\hbar_1 = -0.1, \hbar_2 = -0.1$).

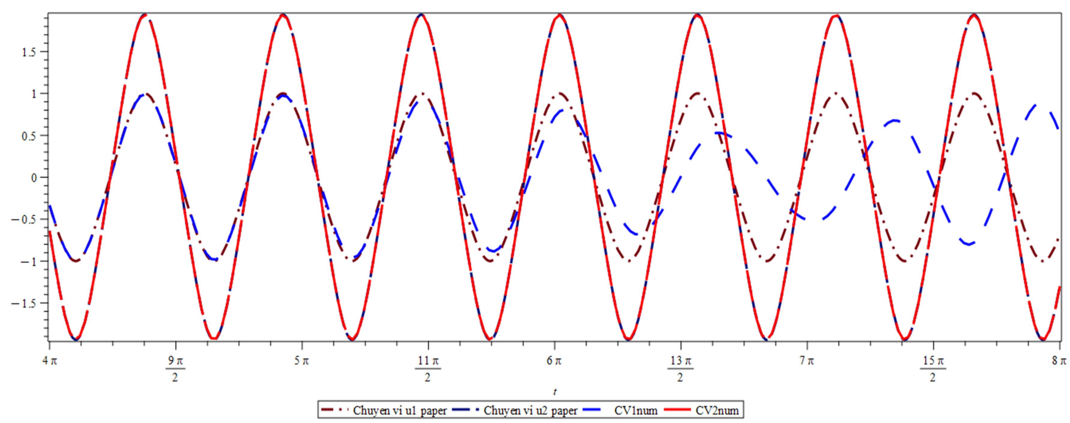

Figure 4. *Values of displacements in the range of t = 4 π - 8π*

Figure 5 reveals that in cases of values $\hbar_1 = -0.13, \hbar_2 = -0.13$ the results of u_1 are not correct with the results calculated by HAM in the region $t > 2\pi$, however u_2 is still correct.

Figure 6 is the results of two parameters δ and ω when changing the auxiliary parameters at the range $\hbar_1 = [-0.2 \rightarrow 0.05], \hbar_2 = [-0.2 \rightarrow 0.05]$. Notice that when in the range $-0.15 < \hbar_1 < 0.04, -0.15 < \hbar_2 < 0.04$, the δ and ω values change slowly, proving that the series converge in that range; When outside the range $\hbar_1, \hbar_2 < -0.15$ *and* $\hbar_1, \hbar_2 > 0.04$ the values change rapidly, proving that the series diverge within that range.

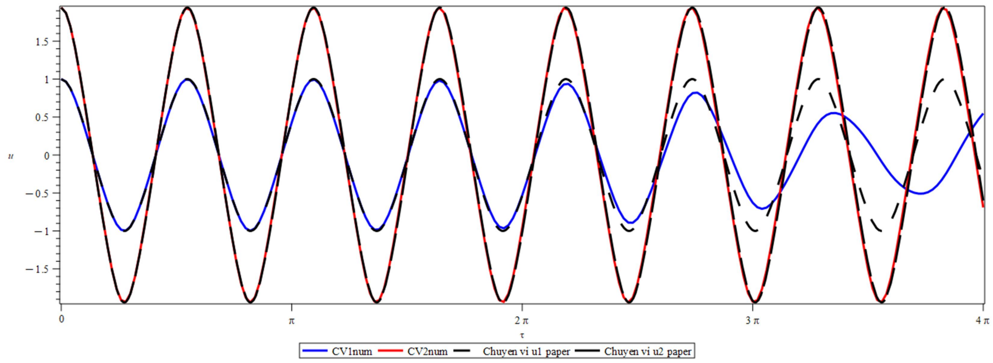

Figure 5. *Values of displacements when values of* $\hbar_1 = -0.13, \hbar_2 = -0.13$

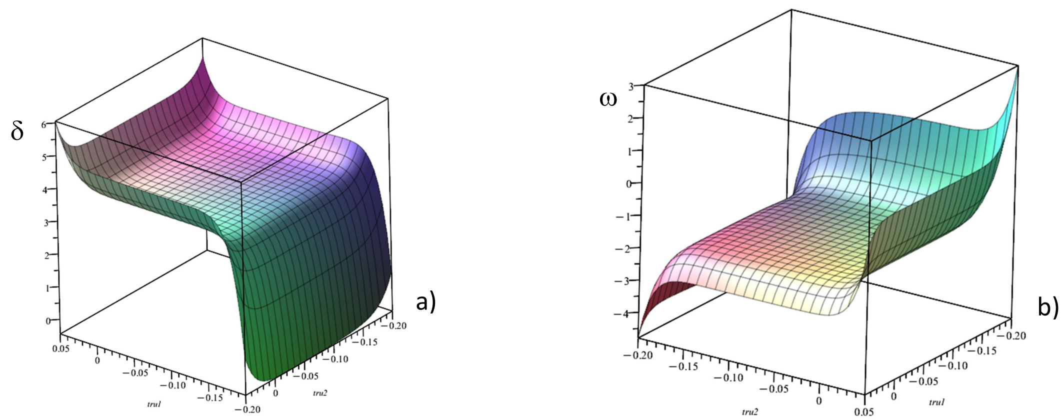

Figure 6. δ*(a) and* ω*(b) values when changing* $\hbar_1 = [-0.2 \to 0.05]$, $\hbar_2 = [-0.2 \to 0.05]$

4. Conclusions

Homotopy analysis method is a useful method when solving nonlinear differential equations. The results obtained in this paper show a reliable displacement value when the auxiliary parameter $\hbar = -0.1$ and a confidence interval when the time variable $t = 0 \to 5.4\pi$. By determining the solution expression as the displacement functions $u_1(t)$ and $u_2(t)$, and receiving the values of the natural oscillation frequencies ω, we can easily analyze the important characteristics of non-linear oscillation, and at the same time proactively survey and analyze the structural model parameters to the quantities that need to be determined. The other methods of nonlinear analysis often depend on a small parameter existing in the original equation, but the homotopy analysis method does not contain any small parameters, so its application to nonlinear problems is very realistic. Analyzing the convergence interval of the solution depending on the auxiliary parameters is a feature of the homotopy analysis method that other methods do not have.

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