q-DIFFERENCE ANALOGUE OF THE LEMMA ON THE LOGARITHMIC DERIVATIVE AND SOME APPLICATIONS

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ABSTRACT

Recently, Nevanlinna theory applied to study difference-differential equations, also value distribution of difference-differential polynomials. This research direction has attracted the attention of many mathematicians in the country as well as around the world. In this paper, by using **q**-difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for meromorphic functions in several variables, we study the proximity function of solutions to **q**-shift difference-partial differential. Our results show that under some suitable conditions of degree of equations, proximity function of solutions is small function in comparing with characteristic functions. In addition, we establish a new lemma on the counting function of zeros of the partial derivative of meromorphic function in several variables, and apply that result to study the value distribution of difference-partial differential polynomials. In our best knowledge, our results are new and some future works can be done by using our previous results.

BỔ ĐỀ q-SAI PHÂN TƯƠNG TỰ CỦA BỔ ĐỀ ĐẠO HÀM LOGARIT VÀ MỘT SỐ ỨNG DỤNG

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TỪ KHÓA

Hàm phân hình nhiều biến
Lý thuyết Nevanlinna
Phương trình **q**-sai phân-dịch chuyển đạo hàm riêng
Phân bố giá trị của đa thức sai phân
Hàm nhỏ

TÓM TẮT

Thời gian gần đây, Lý thuyết Nevanlinna đã được ứng dụng trong nghiên cứu phương trình vi-sai phân, cũng như phân bố giá trị của đa thức đạo hàm-sai phân. Hướng nghiên cứu này đã thu hút được sự quan tâm của nhiều nhà toán học trong và ngoài nước. Trong bài báo này, sử dụng Bổ đề **q**-sai phân tương tự đạo hàm logarit và Lý thuyết Nevanlinna cho hàm phân hình nhiều biến, chúng tôi nghiên cứu hàm xấp xỉ cho nghiệm của phương trình **q**-dịch chuyển sai phân-đạo hàm riêng. Kết quả của chúng tôi chỉ ra rằng với một số điều kiện về bậc của phương trình, hàm xấp xỉ của nghiệm là nhỏ so với hàm đặc trưng. Ngoài ra, bằng việc thiết lập một bổ đề mới về hàm đếm các không điểm của đạo hàm riêng của hàm phân hình nhiều biến, chúng tôi ứng dụng kết quả đó vào nghiên cứu phân bố giá trị của đa thức sai phân-đạo hàm riêng. Theo hiểu biết tốt nhất của chúng tôi, các kết quả trong bài báo là mới và một số nghiên cứu trong tương lai có thể được hoàn thiện bằng việc sử dụng kết quả trước đó của chúng tôi.

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1. Introduction

First, we remind some notations and definitions in Nevanlinna theory for meromorphic functions in several variables. Set $|z|^2=\sum_{j=1}^{m}|z_j|^2$ for all $z=(z_1,...,z_m)\in\mathbb{C}^m$, $S_m(r)=\{z\in\mathbb{C}^m: |z|=r\}$, $\bar{B}_m(r)=\{z\in\mathbb{C}^m: |z|\le r\}, d=\partial+\bar{\partial}, d^c=\frac{1}{4\pi}(\partial-\bar{\partial})$,

$\omega_m=dd^c\log|z|^2, \sigma_m=d^c\log|z|^2\wedge\omega_m{}^{m-1}(z), \nu_m(z)=dd^c|z|^2$. Let ν be divisor in $\mathbb{C}^m$. Set $\operatorname{supp}\nu=\overline{\{z:\nu(z)\neq 0\}}$. We define the counting function of ν by

$N_\nu(r)=\int_1^r\frac{n(t)}{t^{2m-1}}dt, 1<r<+\infty,$ where $n(t)=\int_{\operatorname{supp}\nu\cap\bar{B}_m(t)}\nu_m^{m-1}$, for $m\ge 2$, and $n(t)=\sum_{|z|\le t}\nu$, for $m=1$. Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For a set $\alpha=(\alpha_1,\dots,\alpha_m)$ of nonnegative integers, we set $|\alpha|:=\alpha_1+\dots+\alpha_m$ and $D^{|\alpha|}F:=\frac{\partial^{|\alpha|}}{\partial^{\alpha_1}z_1\dots\partial^{\alpha_m}z_m}$. We define the zero divisor ν_F of F by $\nu_F=\max\{p: D^{|\alpha|}F(z)=0 \text{ for all } \alpha:|\alpha|<p\}$. Let ϕ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $z_0\in\mathbb{C}^m$, the zero divisor ν_ϕ of ϕ is defined as follows: we choose nonzero holomorphic functions F and G on a neighborhood U of z_0 such that $\phi=\frac{F}{G}$ on U and $\dim(F^{-1}(0)\cap G^{-1}(0))\le m-2$, then we put $\nu_\phi=\nu_F$. For each $a\in\mathbb{P}^1(\mathbb{C})$ wih $\phi^{-1}(a)\neq\mathbb{C}^m$, the couting function of a-point of ϕ is definied as following. We denote $\nu_\phi(a)$ by the a-divisor of ϕ. This means, if $\phi=(\phi_0:\phi_1)$ is a expression reducing of ϕ. Then the a-divisor $\nu_\phi(a)$ is the divisor associated with the holomorphic functions $\phi_1-a\phi_0$. Thus $\nu_\phi(a)=\sum_{z\in\mathbb{C}^m}\nu_{\phi_1-a\phi_0}(z)$. We define $n_\phi(r,a)=\int_{\operatorname{supp}\nu_\phi(a)\cap\bar{B}_m(r)}\nu_\phi(a)\nu_m^{m-1}$ outside a set analysis with codimension 2, i.e $\dim((\phi_1-a\phi_0)^{-1}(0)\cap\phi_0^{-1}(0))\le m-2$, for all $m\ge 1$ and $r>0$, where $\operatorname{supp}\nu_\phi(a)$ denotes the closure of the set $\{z\in\mathbb{C}^m:\nu_\phi(a)(z)\neq 0\}$. The *counting function* $N_\phi(r,a)$ (or $N_\phi(r,\frac{1}{\phi-a})$) of a-point of ϕ is defined by $N_\phi(r,a)=\int_1^r\frac{n_\phi(t,a)}{t^{2m-1}}dt.$

For a positive integer M, we define $\bar{n}_\phi^{(M}(r,a)=\int_{\operatorname{supp}\nu_\phi(a)\cap\bar{B}_m(r)}\bar{\nu}_\phi^{(M}(a)\nu_m^{m-1}$ outside a set analysis with codimension 2, i.e $\dim((\phi_1-a\phi_0)^{-1}(0)\cap\phi_0^{-1}(0))\le m-2$, where $\bar{\nu}_\phi^{(M}(a)=0$ if $\nu_\phi(a)<M$ and $\bar{\nu}_\phi^{(M}(a)=1$ if $\nu_\phi(a)\ge M$. The *reduced counting function* $\bar{N}_\phi^{(M}(r,a)$ (or $\bar{N}_\phi^{(M}(r,\frac{1}{\phi-a})$) of a-point of ϕ with multiplicities not less than M is defined by $\bar{N}_\phi^{(M}(r,a)=\int_1^r\frac{\bar{n}_\phi^{(M}(t,a)}{t^{2m-1}}dt$. Let k be a positive integer and $a\in\mathbb{C}$, we set

$$N_k(r,\frac{1}{\phi-a})=\bar{N}_\phi(r,a)+\bar{N}_\phi^{(2}(r,a)+\ldots+\bar{N}_\phi^{(k}(r,a).$$

The *proximity function* of ϕ is defined by

$$m_\phi(r,a)=\begin{cases}\int_{S_m(r)}\log^+\frac{1}{|\phi(z)-a|}\sigma_m(z), a\neq\infty\\ \int_{S_m(r)}\log^+|\phi(z)|\sigma_m(z), a=\infty\end{cases}.$$

The *characteristic function* of ϕ is defined by $T(r,\phi)=m_\phi(r,\infty)+N_\phi(r,\infty)$. The order and hyperorder of ϕ are respectively defined by $\rho(\phi)=\limsup_{r\to\infty}\frac{\log T(r,\phi)}{\log r}$ and $\varsigma(\phi)=\limsup_{r\to\infty}\frac{\log\log T(r,\phi)}{\log r}$. We also denote by $T(r,\frac{1}{\phi-a})=m_\phi(r,a)+N_\phi(r,a)$, where $a\neq\infty$. Some time, we also denote $T_\phi(r,a)$ by $T(r,\frac{1}{\phi-a})$, $m_\phi(r,a)$ by $m(r,\frac{1}{\phi-a})$ and $N_\phi(r,\infty)$ by $N(r,\phi)$. First Main Theorem gives that $T(r,\frac{1}{\phi-a})=T_\phi(r)+O(1)$.

The $\mathbf{q}$-shift difference-partial differential polynomial of meromorphic function f on $\mathbb{C}^m$ is defined by

$$P(z,f)=\sum_{i=1}^{N}\alpha_i(z)\prod_{j=0}^{p}\Big(\frac{\partial^{|I_j|}f}{\partial^{l_{j1}}z_1\ldots\partial^{l_{jm}}z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))\Big)^{S_{ij}},$$

where $T_{\mathbf{q_j},\mathbf{c_j}}(z)=\mathbf{q}_j z+\mathbf{c}_j$, $S_{ij}(1\le i\le N, 0\le j\le p)$, l_{jt}, $1\le t\le m$ are nonnegative integers, M,N are positive integers, $I_j=(l_{j1},\ldots,l_{jm})\in\mathbb{N}^m$, $0\le|I_j|=\sum_{t=1}^{m}l_{jt}\le M$, $\mathbf{c}_j\in\mathbb{C}^m$, $j=0,\ldots,p$, $\mathbf{q}_j=(q_{j1},\ldots,q_{jm})\in\mathbb{C}^m\setminus\{0,1\}$ means that $q_{ji}\neq\{0,1\}, i=1,\ldots,m$, and $\alpha_i(1\le i\le N)$ are small (with respect to f) meromorphic functions. The degree $D(P)$ of f is defined by $D(P)=\max_{1\le i\le N}\{\sum_{j=0}^{p}S_{ij}\}$. We also denote lower degree of P by $d(P)=\min_{1\le i\le N}\{\sum_{j=0}^{p}S_{ij}\}$.

In complex plane, Yang and Laine [1], [2] established some Clunie-type results for difference or q-difference polynomial of meromorphic functions. In 2010, Huang and Chen [3] extended the result of Yang and Laine [1] for q-difference polynomial with many terms of maximal total degree. Cao and Xu [4] gave a difference analogue of Clunie-type lemma for meromorphic functions on $\mathbb{C}^m$ with hyperorder less than 1. In [5], Luong, Nguyen and Pham established a q-difference analogue of the lemma on the logarithmic derivative and apply it to study the value distribution of holomorphic curves. Cao and Korhonen [6] prove a q-difference analogue of the lemma on the logarithmic derivative and using that result, they obtained some q-difference counterpart of Clunie-type results. Hu and Yang [7] and Hao and Zhang [8] extended the Clunie's lemma for meromorphic function in several variables. Motivated by that results, we first prove a Clunie-type result in $\mathbb{C}^m$ as follows:

Theorem 1. *Let* $f:\mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ *be a transcendental meromorphic with order zero such that* $f(0) \neq 0, \infty$. *Assume that* f *is solution in* $\mathbb{C}^m$ *of equation* $f^n(z)P(z,f) = Q(z,f)$, *where* $P(z,f)$, $Q(z,f)$ *are* $\mathbf{q}$*-shift difference-partial differential polynomial of meromorphic function* f, *and the degree* $D(Q)$ *of* $Q(z,f)$ *satisfying* $D(Q) \leq n$, *then we have the following estimate* $m_{P(z,f)}(r,\infty) = S(r,f)$.

Similarly Theorem 1, we have the result as follows.

Theorem 2. *Let* $f:\mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ *be a transcendental meromorphic with order zero such that* $f(0) \neq 0, \infty$. *If* f *is solution in* $\mathbb{C}^m$ *of equation* $f^n(z) = P(z,f)$, *where* $P(z,f)$ *is* $\mathbf{q}$*-shift difference-partial differential polynomial of meromorphic function* f, *and the lower degree* $d(P)$ *of* P *satisfying* $0 \leq n < d(P)$, *then* $m_f(r,0) = S(r,f)$.

In [9], Xu and Zhong study the value distributions of q-shift difference polynomial $f(qz) - af^n(z)$ of a meromorphic function f in complex plane. We denote by $D^k f(z_1,\ldots,z_m) = \dfrac{\partial^k f}{\partial^{l_1} z_1 \ldots \partial^{l_m} z_m}$, where $(l_1,\ldots,l_m) \in \mathbb{N}^m$ and $l_1 + \ldots + l_m = k$. Motivate by the work of Xu and Zhong [9], we establish a result on value distribution of $\mathbf{q}$-shift difference-partial differential polynomial as follows:

Theorem 3. *Let* f *be transcendental meromorphic function on* $\mathbb{C}^m$ *with order zero such that* $f(0) \neq 0, \infty$, *and* $\mathbf{q} \in \mathbb{C}^m \setminus \{0,1\}$, $\mathbf{c} \in \mathbb{C}^m \setminus \{0\}$. *Let* n, k *be two positive integers with* $n \geq 2k+4$. *Then* $f^n(z)(D^k f)(\mathbf{q}z + \mathbf{c}) - a(z)$ *has infinitely zeros, where* $a(z) \not\equiv 0$ *is a small function of* f.

Remark. If $\mathbf{q} = (1,\ldots,1) \in \mathbb{C}^m$, then our theorems still hold for meromorphic functions in several variables with hyperorder less than 1 via Lemma 3 and Lemma 4 which contained in [10].

2. Preliminary

Lemma 1 [5]. *Let* f *be a meromorphic function in* $\mathbb{C}^m$ *of zero order such that* $f(0) \neq 0, \infty$ *and let* $\mathbf{q} = (q_1,\ldots,q_m) \in \mathbb{C}^m \setminus \{0,1\}$. *Then* $m(r, \dfrac{f(\mathbf{q}z)}{f(z)}) = o(T(r,f))$ *on a set with logarithmic density 1.*

Lemma 2 [5]. *Let* f *be a meromorphic function in* $\mathbb{C}^m$ *of zero order such that* $f(0) \neq 0, \infty$, *and let* $\mathbf{q} = (q_1,\ldots,q_m) \in \mathbb{C}^m \setminus \{0,1\}$. *Then* $T(r, f(\mathbf{q}z)) = T(r,f) + S(r,f)$ *a set with logarithmic density 1.*

Lemma 3 [10]. *Let* f *be a non-constant meromorphic function in* $\mathbb{C}^m$ *such that* $f(0) \neq 0, \infty$, *let* $\mathbf{c} \in \mathbb{C}^m$. *If* $\varsigma(f) = \varsigma < 1$, *then* $m(r, \dfrac{f(z+\mathbf{c})}{f(z)}) = S(r,f)$, *for all* $r > 0$ *outside of a possible exceptional set* $E \subset [1, +\infty)$ *of finite logarithmic measure* $lm(E) = \int_E \dfrac{dt}{t} < \infty$.

Note that logarithmic densities of $E \subset [1,+\infty)$ is defined by $\operatorname{logdense(E)}=\lim\limits_{r\to\infty}\dfrac{lm(E)}{\log r}$, then any set with finite logarithmic measure has logarithmic densities zero, and its complement has logarithmic densities 1.

Lemma 4 [10]. *Let $f:\mathbb{C}^m\to\mathbb{P}^1(\mathbb{C})$ be a meromorphic function, let $\mathbf{c}\in\mathbb{C}^m$. If $\varsigma(f)=\varsigma<1$, then $T(r,f(z+\mathbf{c}))=T(r,f)+o(T(r,f))$, where $r\to\infty$ outside of an exceptional set of finite logarithmic measure.*

Combine Lemma 1 - Lemma 3, we get the result as follows:

Lemma 5. *Let f be a meromorphic function in $\mathbb{C}^m$ of zero order such that $f(0)\neq 0,\infty$ and let $\mathbf{q}=(q_1,\ldots,q_m)\in\mathbb{C}^m\setminus\{0,1\},\mathbf{c}\in\mathbb{C}^m$. Then $m(r,\dfrac{f(\mathbf{q}z+\mathbf{c})}{f(z)})=o(T(r,f))$ on a set with logarithmic density 1, and $T(r,f(\mathbf{q}z+\mathbf{c}))=T(r,f)+o(T(r,f))$, where $r\to\infty$ outside of an exceptional set of finite logarithmic measure.*

Lemma 6. *Let f be non-constant meromorphic function on $\mathbb{C}^m$ such that $D^k f\not\equiv 0$, then we have*

$$N(r,\frac{1}{D^k f})\le N(r,\frac{1}{f})+k\bar{N}(r,f)+S(r,f);$$

$$\bar{N}(r,\frac{1}{D^k f})\le N_{k+1}(r,\frac{1}{f})+k\bar{N}(r,f)+S(r,f).$$

Proof. From Logarithmic Derivative lemma and First main theorem, we get

$$T(r,\frac{1}{f})=m(r,\frac{1}{f})+N(r,\frac{1}{f})=m(r,\frac{D^k f}{f}.\frac{1}{D^k f})+N(r,\frac{1}{f})$$

$$\leqslant m(r,\frac{D^k f}{f})+m(r,\frac{1}{D^k f})+N(r,\frac{1}{f})=m(r,\frac{1}{D^k f})+N(r,\frac{1}{f})+S(r,f)$$

$$=T(r,\frac{1}{D^k f})-N(r,\frac{1}{D^k f})+N(r,\frac{1}{f})+S(r,f)$$

$$=T(r,D^k f)-N(r,\frac{1}{D^k f})+N(r,\frac{1}{f})+S(r,f).$$

From a result of Hu and Yang in [11], we have $N(r,D^k f)\le(k+1)\bar{N}(r,f)$. This implies

$$N(r,\frac{1}{D^k f})\leqslant N(r,\frac{1}{f})+T(r,D^k f)-T(r,f)+S(r,f)$$

$$=m(r,D^k f)+N(r,D^k f)-T(r,f)+N(r,\frac{1}{f})+S(r,f).$$

So we have
$$N(r,\frac{1}{D^k f})\leqslant N(r,\frac{1}{f})+k\bar{N}(r,f)+S(r,f). \tag{1}$$

From (1), get

$$\bar{N}(r,\frac{1}{D^k f})+\sum_{p\ge 2}^{+\infty}\bar{N}^{(p}(r,\frac{1}{D^k f})=N(r,\frac{1}{D^k f})\leqslant N(r,\frac{1}{f})+k\bar{N}(r,f)+S(r,f).$$

Therefore, we get

$$\bar{N}(r,\frac{1}{D^k f})\leqslant k\bar{N}(r,f)+(N(r,\frac{1}{f})-\sum_{p\geqslant 2}^{+\infty}\bar{N}^{(p}(r,\frac{1}{D^k f}))+S(r,f). \tag{2}$$

By the definition of $\bar{N}^{(p}(r,\dfrac{1}{D^k f})$, we have $N(r,\dfrac{1}{f})-\sum_{p\geqslant 2}^{+\infty}\bar{N}^{(p}(r,\dfrac{1}{D^k f})\leqslant N_{k+1}(r,\dfrac{1}{f}).$

Combine (1) and (2), we have $\bar{N}(r,\dfrac{1}{D^k f})\le N_{k+1}(r,\dfrac{1}{f})+k\bar{N}(r,f)+S(r,f).$

3. Proof of Theorems

Proof of Theorem 1. We consider two set: $E_1=\{z\in S_m(r):|f(z)|<1\}$, $E_2=S_m(r)\setminus E_1$. Then we have

$$m_{P(z,f)}(r,\infty)=\int_{E_1}\log^+|P(z,f)|\sigma_m(z)+\int_{E_2}\log^+|P(z,f)|\sigma_m(z).$$

We suppose that $P(z,f)$ and $Q(z,f)$ can be written as follows

$$P(z,f)=\sum_{i=1}^{N}\alpha_i(z)\prod_{j=0}^{p}\Big(\frac{\partial^{|I_j|}f}{\partial^{l_{j1}}z_1\ldots\partial^{l_{jm}}z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))\Big)^{S_{ij}},$$

where $S_{ij}(1\le i\le N,0\le j\le p)$, l_{jt}, $1\le t\le m$ are nonnegative integers, M,N are positive integers, $0\le|I_j|=\sum_{t=1}^{m}l_{jt}\le M$, $\mathbf{q}_j\in\mathbb{C}^m\setminus\{0,1\}$, $\mathbf{c}_j\in\mathbb{C}^m$, $j=0,\ldots,p$, and $\alpha_i(1\le i\le N)$ are small (with respect to f) meromorphic functions and

$$Q(z,f)=\sum_{k=1}^{K}\beta_k(z)\prod_{j=0}^{q}\Big(\frac{\partial^{|J_j|}f}{\partial^{h_{j1}}z_1\ldots\partial^{h_{jm}}z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))\Big)^{T_{ij}},$$

where $T_{ij}(1\le k\le K,0\le j\le q)$, h_{jt}, $1\le t\le m$ are nonnegative integers, M,K are positive integers, $0\le|J_j|=\sum_{t=1}^{m}h_{jt}\le M$, and $\beta_k(1\le i\le K)$ are small (with respect to f) meromorphic functions. On E_1, we have

$$|P(z,f)|\le\sum_{i=1}^{N}|\alpha_i(z)|\times\Big|\prod_{j=0}^{p}\Big(\frac{\dfrac{\partial^{|I_j|}f}{\partial^{l_{j1}}z_1\ldots\partial^{l_{jm}}z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}\Big)^{S_{ij}}.\times\Big(\frac{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(z)}\Big)^{S_{ij}}\Big|.$$

Thus, from Logarithmic Derivative lemma for meromorphic function in several variables and Lemma 5, we have

$$\int_{E_1}\log^+|P(z,f)|\sigma_m(z)\le\sum_{i=1}^{N}m_{\alpha_i}(r,\infty)+\sum_{i=1}^{N}\sum_{j=0}^{p}S_{ij}\int_{E_1}\log^+\Big|\frac{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(z)}\Big|\sigma_m(z)$$

$$+\sum_{i=1}^{N}\sum_{j=0}^{p}S_{ij}\int_{E_1}\log^+h_j(z)\sigma_m(z)=S(r,f),\ h_j(z)=\Big|\frac{\dfrac{\partial^{|I_j|}f}{\partial^{l_{j1}}z_1\ldots\partial^{l_{jm}}z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}\Big|. \tag{3}$$

On E_2, we have $P(z,f)=\dfrac{1}{f^n(z)}\sum_{k=1}^{K}\beta_k(z)\mathfrak{h}(z),\ \mathfrak{h}(z)=\prod_{j=0}^{q}\Big(\dfrac{\partial^{|J_j|}f}{\partial^{h_{j1}}z_1\ldots\partial^{h_{jm}}z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))\Big)^{T_{ij}}.$

Thus, we obtain $|P(z,f)|\le\sum_{k=1}^{K}|\beta_k(z)||\mathfrak{h}(z)|,$ where $|\mathfrak{h}(z)|=\mathfrak{h}_1.\mathfrak{h}_2,$ where

$$\mathfrak{h}_1 = \left| \prod_{j=0}^{q} \left(\frac{\frac{\partial^{|J_j|} f}{\partial^{h_{j1}} z_1 \ldots \partial^{h_{jm}} z_m}(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))} \right)^{T_{ij}} \right|, \mathfrak{h}_2 = \left| \prod_{j=0}^{q} \left(\frac{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(z)} \right)^{T_{ij}} \right|.$$

This implies

$$\int_{E_2} \log^+ |P(z,f)| \sigma_m(z) \le \sum_{k=1}^{K} m_{\beta_k}(r,\infty)$$
$$+\int_{E_1} \log^+ \mathfrak{h}_1(z)\sigma_m(z) + \int_{E_2} \log^+ \mathfrak{h}_2(z)\sigma_m(z) \tag{4}$$
$$= S(r,f).$$

Combining (3) and (4), we have $m_{P(z,f)}(r,\infty) = \int_{S_m(r)} \log^+ |P(z,f)| \sigma_m(z) = S(r,f).$

Proof of Theorem 2. We suppose that f is a solution of equation $f^n(z) = P(z,f).$ Since $\sum_{j=0}^{p} S_{ij} \ge d(P)$, $1 \le i \le N,$ then for any z such that $|f(z)| < 1,$ we get

$$\frac{1}{|f(z)|^{d(P)-n}} = \frac{|P(z,f)|}{|f^{d(P)}(z)|} \le \sum_{i=1}^{N} |\alpha_i(z)| \left| \prod_{j=0}^{p} \left(h_j(z) \frac{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(z)} \right)^{S_{ij}} \right|.$$

It implies that

$$(d(P)-n)m_f(r,0) = (d(P)-n)\int_{S_m(r)} \log^+ \frac{1}{|f(z)|} \sigma_m(z) \le \sum_{i=1}^{N} m_{\alpha_i}(r,\infty)$$
$$+\sum_{i=1}^{N}\sum_{j=0}^{p} S_{ij} \int_{E_1} \log^+ h_j(z)\sigma_m(z) + \sum_{i=1}^{N}\sum_{j=0}^{p} S_{ij} \int_{E_1} \log^+ \left| \frac{f(T_{\mathbf{q_j},\mathbf{c_j}}(z))}{f(z)} \right| \sigma_m(z) = S(r,f).$$

Proof of Theorem 3. We denote $T_{\mathbf{q},\mathbf{c}}(z) = \mathbf{q}z + \mathbf{c}$ and set $F = f^n(z)(D^k f)(T_{\mathbf{q},\mathbf{c}}(z)).$ From a result of Hu and Yang in [11], we have $N(r,(D^k f)(T_{\mathbf{q},\mathbf{c}}(z)) \le (k+1)\bar{N}(r, f(T_{\mathbf{q},\mathbf{c}}(z)).$

Therefore, using Lemma 5, we obtain

$$nT(r,f)$$
$$= T\left(r, f^n(D^k f)(T_{\mathbf{q},\mathbf{c}}(z))\frac{1}{(D^k f)(T_{\mathbf{q},\mathbf{c}}(z))}\right) \le T(r,F) + T(r,(D^k f)(T_{\mathbf{q},\mathbf{c}}(z))) + O(1)$$
$$= T(r,F) + m(r,(D^k f)(T_{\mathbf{q},\mathbf{c}}(z))) + N(r,(D^k f)(T_{\mathbf{q},\mathbf{c}}(z)))$$
$$= T(r,F) + m(r, \frac{(D^k f)(T_{\mathbf{q},\mathbf{c}}(z))}{f(T_{\mathbf{q},\mathbf{c}}(z))} f(T_{\mathbf{q},\mathbf{c}}(z))) + N(r,(D^k f)(T_{\mathbf{q},\mathbf{c}}(z)))$$
$$\le T(r,F) + m(r, f(T_{\mathbf{q},\mathbf{c}}(z))) + (k+1)\bar{N}(r, f(T_{\mathbf{q},\mathbf{c}}(z))) + S(r,f).$$

This implies

$$(n-k-1)T(r,f) \le T(r,F) + S(r,f). \tag{5}$$

Apply to Second main theorem for small function and Lemma 6, we obtain the inequality as follows

$$T(r,F) \le \bar{N}(r,F) + \bar{N}(r,\frac{1}{F}) + \bar{N}(r,\frac{1}{F-a}) + S(r,F)$$

$$\le \bar{N}(r,f) + \bar{N}(r,f(T_{\mathbf{q},\mathbf{c}}(z))) + \bar{N}(r,\frac{1}{f}) + \bar{N}(r,\frac{1}{(D^k f)(T_{\mathbf{q},\mathbf{c}}(z))}) + \bar{N}(r,\frac{1}{F-a}) + S(r,f)$$

$$\le 3T(r,f) + N_{k+1}(r,\frac{1}{f(T_{\mathbf{q},\mathbf{c}}(z))}) + k\bar{N}(r,f(T_{\mathbf{q},\mathbf{c}}(z))) + \bar{N}(r,\frac{1}{F-a}) + S(r,f)$$

$$\le (k+4)T(r,f) + \bar{N}(r,\frac{1}{F-a}) + S(r,f).$$

Combining above inequality with (5), we obtain

$$(n-2k-5)T(r,f) \le \bar{N}(r,\frac{1}{F-a}) + S(r,f).$$

From the condittion $n \ge 2k+6,$ we get that $f^n(z)(D^k f)(\mathbf{q}z+\mathbf{c}) - a(z)$ has infinitely zeros.

4. Conclusion

In this paper, we show a $\mathbf{q}$-difference analogue for logarithmic derivative lemma. Apply this result, we study the proximity function of solutions to $\mathbf{q}$-shift difference-partial differential equation and value distribution of difference-polynomials.

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