SOME EXAMPLES ABOUT IRREDUCIBLE DECOMPOSITION OF POWERS OF EDGE IDEALS

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ABSTRACT

In recent years, the construction of algebraic structures on graphs has attracted significant interest from many researchers, especially finding the irreducible decomposition of the powers of an edge ideal. In this paper, we consider $A = Q[x_1, ..., x_k]$ as a polynomial ring in k variables over the field Q, $G = (V, E)$ as a graph with the vertex set $\{x_1,...,x_k\}$ and J_G as the edge ideal associated with G. Our main result is constructing a simple, connected non bipartite graph G over a polynomial ring of 9 variables and calculating the isolated and the embedded irreducible components of powers of the edge ideal J_G^n, with some small number n. In the next research, we can apply this technique to study more the structure of an edge ideal, for examples calculating the indexes astab(J_G) or distab(J_G). It helps us gain a better understanding of the structure of components within graphs, thereby facilitating the development of more efficient algorithms for processing and analyzing graphs.

MỘT SỐ VÍ DỤ VỀ PHÂN TÍCH BẤT KHẢ QUY CỦA LŨY THỪA CỦA IĐÊAN CẠNH

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Đồ thị

TÓM TẮT

Trong những năm gần đây, việc xây dựng cấu trúc đại số trên đồ thị được nhiều nhà khoa học quan tâm nghiên cứu, đặc biệt là việc tìm phân tích bất khả quy của lũy thừa của iđêan cạnh. Trong bài báo này, chúng tôi xét $A = Q[x_1, ..., x_k]$ là vành đa thức k biến trên trường Q, $G = (V, E)$ là đồ thị với tập đỉnh $\{x_1, . . . , x_k\}$ và J_G là iđêan cạnh liên kết với G. Kết quả chính của bài báo là xây dựng một lớp đồ thị đơn, liên thông non-bipartite trên vành đa thức 9 biến và tính toán các thành phần bất khả quy cô lập và thành phần bất khả quy nhúng của một số lũy thừa của iđêan cạnh J_G^n, với n nhỏ. Trong định hướng nghiên cứu tiếp theo, ta có thể dùng kỹ thuật này để nghiên cứu sâu hơn về cấu trúc của iđêan cạnh, chẳng hạn các chỉ số astab(J_G) hoặc distab(J_G). Điều này rất quan trọng, nó giúp ta hiểu rõ hơn về cấu trúc của các thành phần trong đồ thị, từ đó giúp phát triển các thuật toán hiệu quả hơn cho việc xử lý và phân tích đồ thị.

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1. Introduction

Let $G = (V, E)$ be a finite connected simple graph with the vertex set $V = \{u_1, \ldots, u_k\}$ and the edge set E, $A = Q[x_1, ..., x_k]$ the polynomial ring in k variables over a field Q. Recall that the *edge ideal* associated to G, denoted by J_G, is the ideal of R generated by the square-free monomial x_ix_j such that $u_iu_j \in E$. The map $G \mapsto J_G$ gives us a corresponding $1-1$ between the family of graphs and the family of monomial ideals generated by square-free mononials of degree 2. Studying the set of associated prime ideals of powers of edge ideals is considered by many mathematicians (see [1]- [4]). For instance, The authors in [2] proved that the set of associated prime ideals of powers of edge ideals formed a ascending sequence, i. e. $\mathrm{Ass}(A/J^n) \subset \mathrm{Ass}(A/J^{n+1})$, for all n, or Ha et al. [5] described the set of associated prime ideals of powers of edge ideals through graph theory called *generalized ear decomposition.*

We say that a monomial ideal $J \neq A$ is *irreducible* if J cannot be written as a nontrivial intersection of any two monomial ideals of A. A monomial ideal $J \neq A$ has an *irreducible decomposition* (ID for short) if it can be expressed as $J = \cap_{j=1}^{r} J_i$, where $r1$ and each J_i is irreducible. An ID of J is call *irredundant* if any J_i cannot be rejected and the set of all ideals in an irredundant ID of J is denoted by $\mathrm{Irr}(J)$. In the set $\mathrm{Irr}(J)$, each element is called *irreducible component* (IC for short), and the IC is called *isolated* if its radical does not contain radicals of any other irreducible components (ICs for short), otherwise it is called *embedded*.

For edge ideals-a class of special monomial ideals, the finding ID of powers of edge ideals is always an interesting problem (see [6] - [9]). Recently, Morales et al. [10] have described the set of ICs of powers of edge ideals $\mathrm{Irr}(J_G^n)$ by using some tools from graph theory such as the ear decomposition theorem and the structure of graph theorem. More concretely, they described the general form of ICs of $\mathrm{Irr}(J_G^n)$, gave the formula to compute the number of isolated ICs and the way to compute isolated and embedded ICs. The purpose of our paper is to construct a special graph with 9 vertexes over a 9 variable-polynomial ring $Q[a, \ldots, i]$ and then compute isolated and embedded ICs of some small powers of edge ideal J_G in order to show how do Theorem 3.1 and Theorem 5.3 in [10] operate.

In the next section, we will recall some results about ID, eardecomposition, replication graphs,... In the section 3, we will compute isolated and embedded ICs of $\mathrm{Irr}(J_G^n)$ for some small number $n \in \mathbb{N}$ of a special graph (see Example 3.2 and Example 3.4).

2. Preliminaries

In this section, we recall some terminologies that will be used in the rest of the paper. For a non-zero vector $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$, we set $\mathbf{a} + \mathbf{1} = (a_1 + 1, \ldots, a_k + 1) \in \mathbb{N}^k$, $\mathfrak{m}^{\mathbf{a}} := (x_i^{a_i} \mid i = 1, \ldots, k,\ a_i > 0)$, $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_k^{a_k}$ and $\mathrm{Supp}(\mathbf{a}) = \mathrm{Supp}(\mathbf{x}^{\mathbf{a}}) := \{x_i \in V(G) \mid a_i \neq 0\}$.

It is clear that a non-zero monomial ideal J of A is irreducible if J is of the form $\mathfrak{m}^{\mathbf{c}}$ for some non-zero vector $\mathbf{c} \in \mathbb{N}^k$ and its ID is an expression of the form $I = \mathfrak{m}^{\mathbf{c_1}} \cap \ldots \cap \mathfrak{m}^{\mathbf{c_r}}$, for some non-zero vectors $\mathbf{c_1}, \ldots, \mathbf{c_r} \in \mathbb{N}^k$.

Definition 2.1 (i) A graph G is called *factor-critical* if for any vertex v in G the graph $G - v$ has a perfect matching. A set $U \subset V$ is called *factor-critical* if the induced subgraph on U is factor-critical. A set $K \subset V$ is called *matching-critical* if the induced subgraph on K is a disjoint union of factor-critical graphs.

(ii) A set $X \subset V$ is a *clique* of G if the induced subgraph $G[X]$ is a complete graph and it is a *coclique* if the induced subgraph $G[X]$ has no edges.

(iii) *An ear decomposition* $G_0, G_1, \ldots, G_r = G$ of a graph G is a sequence of graphs with the first graph G_0 being a vertex, edge, even cycle, or odd cycle, and each graph G_{i+1} is obtained from G_i by adding an ear.

(iv) Let $\mathbf{a} \in \mathbb{N}^k$ be a non zero vector and we set $P_i = \{x_i = x_i^{(1)}, \ldots, x_i^{(a_i)}\}$ for each $a_i > 0$. The graph $P := p_{\mathbf{a}}(G)$ with the vertex set $V(P) = \cup_{a_i>0} P_i$ and the edge set $E(P) = \{x_i^{(l)} x_j^{(m)} \mid x_i \in P_i, x_j \in P_j, x_i x_j \in E\}$ is called *the replication* of G by the vector $\mathbf{a}$. The *support* of P is the set $\mathrm{Supp}(P) := V(P) \cap V = \mathrm{Supp}(\mathbf{a})$, we denote $N_G(P) = N(V(P) \cap V)$. For small values of $a_i \leq 3$ sometimes we will write x_i, x_i', x_i'' instead of $x_i^{(1)}, x_i^{(2)}, x_i^{(3)}$.

From now on, for each subset X belongs to V, we define $N(X)$ be the set of vertices that are adjacent to some elements in X.

Remark 2.2 (i) A set $X \subset V$ is a coclique iff $N(X) \cap X = \emptyset$ and X is a maximal coclique iff $V = N(X) \cup X$.

(ii) It is easy to see that odd cycles are factor-critical, in particular the set of one element is factor-critical; If X is factor-critical, then the number of vertices of X is odd and $\nu(X) = \frac{\sharp(X)-1}{2}$, in particular complete odd graphs are factor-critical.

3. Isolated and embedded irreducible components of $\mathrm{Irr}(I_G{}^k)$

The following theorem allows us to find non embedded ICs of $\mathrm{Irr}(J_G^n)$.

Theorem 3.1 [10, Theorem 3.1] *(i) Let $X \subset V$ be a maximal coclique, $Y := V \setminus X$ and M a monomial. Then $M\mathbf{x}^{n\mathbf{1}_X}$ is a corner element of $J_G^n + \mathfrak{m}^{(n+1)\mathbf{1}_V}$ if and only if M is a monomial of degree $n-1$ with support in Y.*

(ii) Every non embedded IC of J_G^n can be written as $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_X}$ for some set $X \subset V$ such that $\mathrm{Supp}(\mathbf{a}) \subset X$, *$Y := V \setminus X$ is a maximal coclique inside V and $M := \mathbf{x}^{\mathbf{a}}$ is a monomial of degree $n-1$.*

(iii) Let $X_1, \ldots, X_\rho$ be the maximal coclique sets inside V and $\mu_i = d - \sharp X_i$. Then the number of non embedded ICs of J_G^n is exactly $\sum_{i=1}^{\rho} \binom{\mu_i - 1 + n - 1}{\mu_i - 1}$, it coincides with a polynomial of degree $\mathrm{bight}(J_G) - 1$, *where* $\mathrm{bight}(J_G)$ *is the biggest height of ICs of J_G.*

Example 3.2 Let $G = (V, E)$ be the graph with the vertex set $V = \{a, \ldots, i\}$, $A = Q[a, \ldots, i]$ the polynomial ring of 9 variables $a, \ldots, i$ over the field Q and J_G the edge ideal associated with the graph G. Find all isolated ICs of $\mathrm{Irr}(J_G^n)$ with $n = 1, 2$ (see Figure 1)?

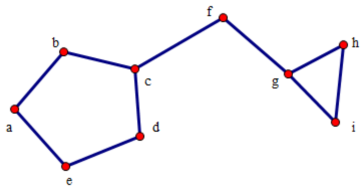

Figure 1. *Example 3.2*

It is known that the maximal ideal of A is $\mathfrak{m} = (a, b, c, d, e, f, g, h, i)R$ and edge ideal associated with the graph G is

$$J_G = \{ab, bc, cd, de, ea, cf, fg, gh, gi, hi\}.$$

By Theorem 3.1, we should do several steps:

(1) Firstly, find all the minimal vertex covers of G, they are the following 15 sets:

$$\begin{aligned}
&X_1 = \{a, b, d, f, h, i\}; X_2 = \{a, b, d, f, g, h\}; X_3 = \{a, b, d, f, g, i\};\\
&X_4 = \{a, c, e, g, h\}; X_5 = \{a, c, e, g, i\}; X_6 = \{a, c, e, f, h, i\};\\
&X_7 = \{a, c, d, g, h\}; X_8 = \{a, c, d, g, i\}; X_9 = \{a, c, d, f, h, i\};\\
&X_{10} = \{b, c, e, g, h\}; X_{11} = \{b, c, e, g, i\}; X_{12} = \{b, c, e, f, h, i\};\\
&X_{13} = \{b, d, e, f, h, i\}; X_{14} = \{b, d, e, f, g, h\}; X_{15} = \{b, d, e, f, g, i\}.
\end{aligned}$$

(2) Secondly, the number of isolated ICs of $\mathrm{Irr}(J_G^n)$ can be computed by Theorem 3.1, (iii). Since we have 9 sets X_i with 6 elements and 6 sets X_i with 5 elements:

- For $n = 1$, the results as below:

$$\sum_{i=1}^{15}\binom{\mu_i - 1 + k - 1}{\mu_i - 1} = 9\binom{6-1+1-1}{6-1} + 6\binom{5-1+1-1}{5-1} = 9\binom{5}{5} + 6\binom{4}{4} = 15.$$

- For $n = 2$, the results as below:

$$\sum_{i=1}^{15}\binom{\mu_i - 1 + k - 1}{\mu_i - 1} = 9\binom{6-1+2-1}{6-1} + 6\binom{5-1+2-1}{5-1} = 9\binom{6}{5} + 6\binom{5}{4} = 84.$$

(3) Thirdly, we can find concretely all isolated ICs of $\mathrm{Irr}(J_G^n)$ by Theorem 3.1, (ii).

- The case $n = 1$: since all monomials of degree $n - 1 = 0$, all isolated ICs have the form $\mathfrak{m}^{1_{X_i}}$, for $i = 1, \ldots, 15$. Thus
$$J_G = \{\mathfrak{m}^{1_{X_i}} \mid i = 1, \ldots, 15\}.$$

- The case $n = 2$: We consider all monomials $M = \mathbf{x}^{\mathbf{a}}$ of degree $n - 1 = 1$ such that $\mathrm{Supp}(\mathbf{a}) \subset X_i$.

  - For instance, we find all isolated ICs with respect to the set $X_4 = \{a, c, e, g, h\}$. We can see that $\mathbf{1}_{X_4} = (1, 0, 1, 0, 1, 0, 1, 1, 0)$, and the monomials of degree 1 whose support belong to X_4 are a, c, e, g, h.

  + Because the monomial a has vecto $\mathbf{a} = (1, 0, \cdots, 0)$, so the isolated IC is $\mathfrak{m}^{\mathbf{a}+1_{X_4}} = (a^2, c, e, g, h)R$.

  + Similarly, we also have isolated ICs with respect to the monomial c, e, g, h are:

$$(a, c^2, e, g, h)A, (a, c, e^2, g, h)A, (a, c, e, g^2, h)A, (a, c, e, g, h^2)A.$$

  - Continue to do similarly with the sets $X_1, X_2, X_3, X_5, \ldots, X_{15}$, we can find 84 isolated ICs of $\mathrm{Irr}(J_G^2)$.

It is much more difficult for us to find embedded ICs of J_G^n, even n is a small number, but the following theorem is an useful tool using replication technique.

Theorem 3.3 [10, Theorem 5.3] *Let $n2$ be an integer, $\mathbf{a} \in \mathbb{N}^k$ be a nonzero vector, $X \subset V$ such that $\mathrm{Supp}(\mathbf{a}) \subset X$. Let denote $Y := V \setminus X$ and $P := p_{\mathbf{a}}(G)$ the replication of G by $\mathbf{a}$. Assume that $\mathrm{Supp}(\mathbf{a}) \cap N(X) = \emptyset$. Then the following statements are equivalent:*
(i) $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_X}$ is an embedded IC of J_G^n.
(ii) The sets P and X satisfy the following properties:

1. *X is a coclique set, $\nu(P) = n - 1$ and $V = N_G(D(P)) \cup X \cup N(X)$.*
2. *$C(P) = \emptyset$, i.e. $P = D(P) \cup A(P)$ in the Gallai-Edmonds's canonical decomposition.*

Now we can apply [10, Theorem 5.3] for finding embedded ICs of $\mathrm{Irr}(J_G^n)$ in Example 3.2.

Example 3.4 Find embedded ICs of $\mathrm{Irr}(J_G^n)$ in Example 3.2 for the cases $n = 2, 3, 4$.

By Theorem 3.3, we first need to find factor-critical set P such that $\nu(P) = n-1$, a vecto $\mathbf{a}$ and the set X such that $\mathrm{Supp}(\mathbf{a}) \cap N(X) = \emptyset, \mathrm{Supp}(\mathbf{a}) \subset Y = V \setminus X$ and satisfies the property Theorem 3.3,(ii), i.e. X is coclique set, $V = N_G(D(P)) \cup Z \cup N(X)$, $P = D(P) \cup A(P)$ within the Gallai-Edmonds canonical decomposition.

(i) The case $n = 2$.

+ In this graph, there is only one factor-critical set $P = \{g, h, i\}$ satisfying $\nu(P) = n - 1 = 2 - 1 = 1$.

+ Choose vecto $\mathbf{a} = (0, 0, 0, 0, 0, 0, 1, 1, 1) \in \mathbb{N}^9$.

+ We find all the sets X such that $N(X) \cap \mathrm{Supp}(\mathbf{a}) = \emptyset$. Then we have

$$X_1 = \{a, c\}, X_2 = \{a, d\}, X_3 = \{b, d\}, X_4 = \{b, e\}, X_5 = \{c, e\}.$$

+ We can see that $P = \{g, h, i\}$ and $X_1 = \{a, c\}$ satisfy the condition (ii) of Theorem 3.3, i.e.

(1) X_1 is coclique set, $\nu(P) = 2 - 1 = 1$ and since $N(X_1) = \{b, e, d, f\}$, $N(D(P)) = \{f, g, h, i\}$ we have $V = N(D(P)) \cup X_1 \cup N(X_1)$.

(2) Since $D(P) = P$, we have $A(P) = C(P) = \emptyset$, so $P = D(P) \cup A(P)$ in the Gallai-Edmonds's canonical decomposition.

Let consider $Y_1 = V \setminus X_1 = \{b, d, e, f, g, h, i\}$. Since $\mathbf{a} + \mathbf{1}_{Y_1} = (0, 1, 0, 1, 1, 1, 2, 2, 2)$, it implies that $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_1}} = (b, d, e, f, g^2, h^2, i^2)$ is embedded IC of J_G^2.

Do similarly to the sets $X_2, \ldots, X_5$, we can find embedded ICs J_G^2:

$$\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_2}} = (b, c, e, f, g^2, h^2, i^2); \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_3}} = (a, c, e, f, g^2, h^2, i^2)$$
$$\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_4}} = (a, c, d, f, g^2, h^2, i^2); \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_5}} = (a, b, d, f, g^2, h^2, i^2).$$

(ii) The case $k = 3$.

• Choose the factor-critical set $P = \{a, b, c, d, e\}$ satisfying the condition $\nu(P) = n - 1 = 3 - 1 = 2$.

+ Choose vecto $\mathbf{a} = (1, 1, 1, 1, 1, 0, 0, 0, 0) \in \mathbb{N}^9$.

+ We have 3 sets X such that $N(X) \cap \mathrm{Supp}(\mathbf{a}) = \emptyset$. They are the sets

$$X_1 = \{g\}, X_2 = \{h\}, X_3 = \{i\}.$$

+ It is clear that the sets $P = \{a, b, c, d, e\}$ and $X_1 = \{g\}$ satisfy the condition (ii) of Theorem 3.3:

(1) X_1 is coclique set, $\nu(P) = 3 - 1 = 2$ and since $N(X_1) = \{f, h, i\}$, $N(D(P)) = \{a, b, c, d, e, f\}$ we have $V = N(D(P)) \cup X_1 \cup N(X_1)$.

(2) Since $D(P) = P$, $A(P) = C(P) = \emptyset$. Thus $P = D(P) \cup A(P)$.

Now we take the set $Y_1 = V \setminus X_1 = \{a, b, c, d, e, f, h, i\}$. It is clear that $\mathbf{a} + \mathbf{1}_{Y_1} = (2, 2, 2, 2, 2, 1, 0, 1, 1)$ so $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_1}} = (a^2, b^2, c^2, d^2, e^2, f, h, i)$ is the embedded irreducible component of J_G^3.

Do similarly with the sets X_2, X_3 we also have

$$\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_2}} = (a^2, b^2, c^2, d^2, e^2, f, g, i); \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_3}} = (a^2, b^2, c^2, d^2, e^2, f, g, h)$$

belong to $\mathrm{Irr}(J_G^3)$.

• Replicate the vertex g of the triangle $\{g, h, i\}$, we have that $gf, fg', g'h$ is an ear of $\{g, h, i\}$. Choose the pentagon $P = \{g, f, g', h, i\}$ that is the factor-critical satisfying the condition $\nu(S) = n - 1 = 3 - 1 = 2$. (see Figure 2).

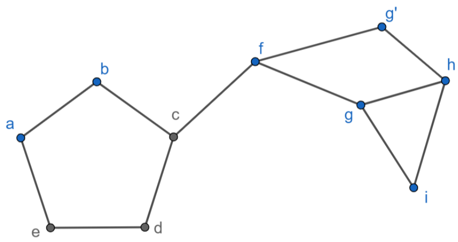

Figure 2. *Example 3.4 when replicating the vertex g*

+ Choose vecto $\mathbf{a} = (0, 0, 0, 0, 0, 1, 2, 1, 1) \in \mathbb{N}^9$.

+ Then we can see there are 3 sets X such that $N(X) \cap \mathrm{Supp}(\mathbf{a}) = \emptyset$:

$$X_1 = \{a, d\}, X_2 = \{b, d\}, X_3 = \{b, e\}.$$

+ Check the condition (ii) of Theorem 3.3, we have

(1) It is clear that X_1 is the coclique set and

$$N(D(P)) \cup X_1 \cup N(X_1) = \{g, f, g', h, i, c\} \cup \{a, c\} \cup \{b, c, e\} = V.$$

(2) Since $D(P) = P$, we have $A(P) = C(P) = \emptyset$, so $P = D(P) \cup A(P)$.

Now take the set $Y_1 = V \setminus X_1 = \{b, c, e, f, h, i\}$. It can be implied from $\mathbf{a} + \mathbf{1}_{Y_1} = (0, 1, 1, 0, 1, 2, 3, 2, 2)$ that $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_1}} = (b, c, e, f^2, g^3, h^2, i^2)$ is the embedded IC of J_G^3.

Do similarly with the sets X_2, X_3 we can find

$$\mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_2}} = (a, c, e, f^2, g^3, h^2, i^2); \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_3}} = (a, c, d, f^2, g^3, h^2, i^2)$$

belong to $\mathrm{Irr}(J_G^3)$.

• We can also replicate 2 vertexes h, i of the triangle $\{g, h, i\}$. Then $hi', i'h', h'i$ is an ear of $\{g, h, i\}$. Now we choose the pentagon $P = \{g, h, i', h', i\}$ that is the factor-critical satisfying $\nu(P) = n - 1 = 3 - 1 = 2$. (see Figure 3)

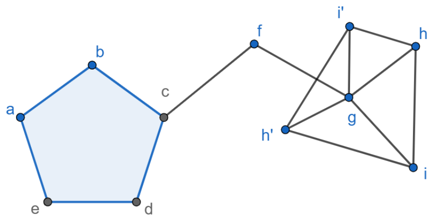

Figure 3. *Example 3.4 when replicating h, i*

+ Choose vecto $\mathbf{a} = (0, 0, 0, 0, 0, 0, 1, 2, 2) \in \mathbb{N}^9$.

+ With the similar method we can find the 5 sets X

$$X_1 = \{a, c\}, X_2 = \{a, d\}, X_3 = \{b, d\}, X_4 = \{b, e\}, X_5 = \{c, e\}$$

such that $N(X) \cap \mathrm{Supp}(\mathbf{a}) = \emptyset$ and satisfying the condition (ii) of Theorem 3.3. So we can find 5 ICs of $\mathrm{Irr}(J_G^3)$:

$$\begin{aligned}
Y_1 &= \{b, d, e, f, g, h, i\} \Rightarrow \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_1}} = (b, d, e, f, g^2, h^3, i^3);\\
Y_2 &= \{b, c, e, f, g, h, i\} \Rightarrow \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_2}} = (b, c, e, f, g^2, h^3, i^3);\\
Y_3 &= \{a, c, e, f, g, h, i\} \Rightarrow \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_3}} = (a, c, e, f, g^2, h^3, i^3);\\
Y_4 &= \{a, c, d, f, g, h, i\} \Rightarrow \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_4}} = (a, c, d, f, g^2, h^3, i^3);\\
Y_5 &= \{a, b, d, f, g, h, i\} \Rightarrow \mathfrak{m}^{\mathbf{a}+\mathbf{1}_{Y_5}} = (a, b, d, f, g^2, h^3, i^3)
\end{aligned}$$

(iii) The case $n = 4$.

For this case, similar to the case $n = 3$, we need to find the factor-critical set P such that $\nu(P) = 3$ and satisfies the conditions of Theorem 3.3. In order to do that, we need the following replicating technique.

• Replicate the vertexes of the pentagon $\{a, b, c, d, e\}$, for instance a', e' (or a', c' or b', c' or c', d' or d', e'), we have some factor-critical sets P satisfying $\nu(P) = 3$.

• Replicate two edges of the triangle $\{g, h, i\}$, for instance h', i' and g', i'', we have the sequence of ears

$$G_0 = \{g, h, i\}, G_1 = \{g, h', i, g'\}, G_2 = \{h, g', i'', h'\}.$$

We also get that $P = \{g, h, i, h', i', g', i''\}$ is factor-critical set with $\nu(P) = 3$.

After that by similar technique we find vecto **a**, the coclique sets X, the sets $Y = V \setminus X$ and finally we can find the embedded ICs with respect to these sets.

4. Conclusion

The paper has constructed and presented some examples of ID of powers of edge ideals on polynomial rings through computing the isolated and embedded ICs of J_G^n. Through Example 3.2 and Example 3.4, we can see that the results in [10] are very useful to compute isolated and embedded ICs of the powers of many classes of edge ideals. This is a very important result for our future research. Specifically, we will use these techniques to compute $\mathrm{astab}(J_G)$ or $\mathrm{distab}(J_G)$.

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