A RELAXED INERTIAL METHOD FOR SOLVING THE SPLIT FEASIBILITY PROBLEM IN HILBERT SPACES

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ARTICLE INFO

Received: 20/01/2025
Revised: 08/4/2025
Published: 12/4/2025

KEYWORDS

Split feasibility problem
Hilbert spaces
Inertial method
Relaxation method
Self-adaptive

ABSTRACT

Since its introduction in 1994, the split feasibility problem has found numerous practical applications in various fields, such as digital engineering and medicine. In this paper, we investigate and address the split feasibility problem in real Hilbert spaces. We propose a novel method that leverages the inertial technique combined with a relaxation technique with a self- adaptive step size criterion to approximate the solution of the problem. The proposed method achieves strong convergence under some conditions on the control parameters and without prior knowledge of the transformation operators and the monotone and Lipschitz continuous constants of the involved operators. Additionally, we apply our algorithm to image restoration problems, comparing the effectiveness of the new method with the methods being compared. The experimental results validate that the proposed method not only ensures feasibility but also enhances efficiency in solving image restoration problems. This confirms the wide applicability and high practicality of the method we have developed.

MỘT PHƯƠNG PHÁP NỚI LỎNG GIẢI BÀI TOÁN CHẤP NHẬN TÁCH TRONG KHÔNG GIAN HILBERT

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THÔNG TIN BÀI BÁO

Ngày nhận bài: 20/01/2025
Ngày hoàn thiện: 08/4/2025
Ngày đăng: 12/4/2025

TỪ KHÓA

Bài toán chấp nhận tách
Không gian Hilbert
Phương pháp quán tính
Phương pháp nới lỏng
Tự thích nghi

TÓM TẮT

Từ khi được giới thiệu vào năm 1994, bài toán chấp nhận tách đã có những ứng dụng đáng kể trong các lĩnh vực như kỹ thuật số và y học. Trong bài báo này, chúng tôi nghiên cứu và giải quyết bài toán chấp nhận tách trong không gian Hilbert thực. Chúng tôi đề xuất một phương pháp mới sử dụng kỹ thuật quán tính kết hợp với kỹ thuật nới lỏng với tiêu chí kích thước bước tự thích nghi để xấp xỉ nghiệm của bài toán. Phương pháp đề xuất đạt được sự hội tụ mạnh dưới một số điều kiện về các tham số điều khiển và không yêu cầu thông tin trước về các toán tử chuyển đổi hoặc các hằng số đơn điệu và liên tục Lipschitz của các toán tử liên quan. Thêm vào đó, chúng tôi áp dụng thuật toán của mình vào các bài toán khôi phục ảnh, so sánh hiệu quả của phương pháp mới với các phương pháp đang được so sánh. Các kết quả thí nghiệm xác nhận rằng phương pháp đề xuất không chỉ đảm bảo tính khả thi mà còn nâng cao hiệu quả trong việc giải quyết các bài toán khôi phục ảnh. Điều này khẳng định tính ứng dụng rộng rãi và tính thực tiễn cao của phương pháp mà chúng tôi đã phát triển.

DOI: https://doi.org/10.34238/tnu-jst.11923

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1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let $C_1 \subseteq H_1$ and $C_2 \subseteq H_2$ be nonempty closed convex subsets, and let $F_2 : H_1 \to H_2$ be a bounded linear mapping. The split feasibility problem (SFP), proposed by Censor and Elfving [1] in 1994, is stated as follows:

$$\text{Find } x^* \in C_1 \text{ such that } F_2 x^* \in C_2. \tag{SFP}$$

Then, $\Gamma := \{x^* \in C_1 \mid F_2 x^* \in C_2\}$ is the solution set of the SFP. The SFP has garnered significant scholarly interest due to its extensive practical applications, including image restoration, compressed sensing, and intensity-modulated radiation therapy (see [2]- [4]).

To solve the SFP, Byrne [5] proposed the CQ algorithm for the first time infinite dimensional spaces. The algorithm starts with an initial guess x^0 and generates x^{k+1} through iterative steps:

$$x^{k+1} = P_{C_1}(I - \lambda F_2^*(I - P_{C_2})F_2)x^k, \tag{CQ}$$

where F_2^* denotes the transpose of F_2. The weak convergence of the CQ method is guaranteed under the assumption that $\lambda \in \left(0, \frac{2}{L}\right)$ with L being the largest eigenvalue of the matrix $F_2^* F_2$.

The CQ method has been extended by several authors in order to solve the (SFP). For instance, the improved method by López et al. [6], Nesterov [7] and Wang [8]. To remove the constraint requirement on λ and achieve strong convergence, Nguyen et al. [9] proposed an inertial iterative algorithm that combines the one-step inertial method with self-adaptive step size:

$$\begin{cases} w^k = x^k + \theta_k \left(x^k - x^{k-1}\right), \\ x^{k+1} = \alpha_k f\left(x^k\right) + (1-\alpha_k)\left[(1-\eta)I + \eta P_{C_1}\right]\left[I - \gamma_k F_2^*(I - P_{C_2})F_2\right] w^k, \end{cases} \tag{1}$$

where $f : H_1 \to H_1$ be a contraction mapping with $\tau \in [0,1)$ as the contraction coefficient, $\{\alpha_k\}$ is a sequence of positive real numbers in $(0,1)$ satisfying $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\eta \in (0,2)$, γ_k is the self-adaptive step size and $\theta_k(x^k - x^{k-1})$ is the inertial iteration step with θ_k such that:

$$\gamma_k = \rho_k \frac{\|(I - P_{C_2})F_2 w^k\|^2}{\|F_2^*(I - P_{C_2})F_2 w^k\|^2 + e_k}, \tag{γ_k}$$

$$\theta_k = \begin{cases} \min\left\{\frac{\eta_k}{\|x^k - x^{k-1}\|}, \theta\right\}, & \text{if } x^k \neq x^{k-1}, \\ \theta, & \text{otherwise,} \end{cases} \tag{θ_k}$$

where $\theta \in (0,1)$ is a constant, $\{\eta_k\}$ is a positive sequence such that $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$, $\{\rho_k\} \subset (0,2)$ and $\{e_k\} \subset (0,\infty)$.

In the present article, we develop a new method for approximating the solution of the (SFP) in real Hilbert spaces by using the inertial technique combined with the relaxation technique with a self-adaptive step size criterion. As a result, our method ensures the strong convergence of the generated sequences, does not require the norm of operators, the monotone and Lipschitz continuous coefficients of the involved operators and enhances convergence speed. An application of the SFP in the image recovery problem is performed to compare our algorithm with the CQ algorithm of Byrne [5] and the inertial iterative algorithm of Nguyen et al. [9], supporting these claims.

2. Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let H be a real Hilbert space with inner product $\langle . \rangle$ and norm $\|.\|$, and C be a nonempty, closed and convex subset of H. In what follows, we write $x^k \rightharpoonup x$ indicates that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicates that the sequence $\{x^k\}$ converges strongly to x. For all $x,y \in H$ and $\lambda \in (0,1)$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \text{ and } \|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

A mapping $T : H \to H$ is called to be ℓ-Lipschitz continuous on H, if there exists a constant $\ell > 0$ such that $\|Tx - Ty\| \leq \ell\|x-y\|$, $\forall x,y \in H$; Contraction with coefficient $\ell \in (0,1)$; Nonexpansive with $\ell = 1$; Monotone, if $\langle Tx - Ty, x-y\rangle \geq 0$, $\forall x,y \in H$; β-strongly monotone, if there exists a constant $\beta > 0$ such that $\langle Tx - Ty, x - y\rangle \geq \beta\|x-y\|^2$, $\forall x,y \in H$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, which satisfies $\|x - P_C x\| \leq \|x-y\|$, $\forall y \in C$. Hence, $P_C : H \to C$ is called the metric projection of H onto C.

Lemma 2.1 (see [10])**.** Let $C \subseteq H$ be a nonempty closed convex subset. For all $x \in H$ and $y \in C$, then (1): $\langle x - P_C x, y - P_C x\rangle \leq 0$ and (2): $\|P_C x - y\|^2 \leq \|x-y\|^2 - \|x - P_C x\|^2$.

Lemma 2.2 (see [11])**.** Let H be a real Hilbert space. Suppose that $A : H \to H$ is ℓ-Lipschitz and η-strongly monotone over a closed and convex subset $\Omega \subset H$. Then $x^* \in \Omega$ is a unique solution of variational inequality problem (VIP(A,Ω)): Find $x^* \in \Omega$ such that $\langle Ax^*, x - x^*\rangle \geq 0$, $\forall x \in \Omega$.

Lemma 2.3 (see [12])**.** Suppose $\{a_k\}$ and $\{b_k\}$ are two non-negative real sequences such that $a_{k+1} \leq a_k + b_k$ for all $k \geq 1$. If $\sum_{k=1}^{\infty} b_k < +\infty$, then $\lim_{k\to\infty} a_k$ exists.

Lemma 2.4 (see [13])**.** Assume that $A : H \to H$ is a continuous and monotone operator. Then x^* is a solution of VIP: Find $x^* \in \Omega$ such that $\langle Ax^*, x - x^*\rangle \geq 0$, $\forall x \in \Omega$, if and only if x^* is a solution of following problem: Find $x^* \in \Omega$ such that $\langle Ax, x - x^*\rangle \geq 0$, $\forall x \in \Omega$.

Lemma 2.5 (see [14])**.** Let $\{\alpha_k\}$ be a sequence in $(0,1)$ with $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\{s_k\}$ be a sequence of non-negative real numbers and $\{d_k\}$ be a sequence of real numbers. Assume that, for all $k \geq 1$, $s_{k+1} \leq (1-\alpha_k)s_k + \alpha_k d_k$. If $\limsup_{l\to\infty} d_{k_l} \leq 0$, for every subsequence $\{s_{k_l}\}$ of $\{s_k\}$ satisfying $\liminf_{l\to\infty}(s_{k_l+1} - s_{k_l}) \geq 0$, then $\lim_{k\to\infty} s_k = 0$.

3. Main results

3.1 Algorithm and convergence

In this section, let H_1 and H_2 be real Hilbert spaces, $C_1 \subseteq H_1$ and $C_2 \subseteq H_2$ be nonempty closed convex subsets, the mapping $A : H_1 \to H_1$ be η-strongly monotone and ℓ-Lipschitz continuous, $F_1 : H_1 \to H_1$ be the identity operator, and $F_2 : H_1 \to H_2$ be a bounded linear operator. We investigate the (SFP) under the following conditions.

Assumption 3.1. For all $j = 1,2$, we have **(C1)**: $\{\alpha_k\} \subset (0,1)$, $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$, **(C2)**: $\{\eta_k\} \subset (0,\infty)$ such that $\lim_{k\to\infty} \eta_k/\alpha_k = 0$, **(C3)**: $\{\xi_k\} \subset [a,b] \subset (0,1)$, **(C4)**: $0 < b_j < \overline{b}_j < 2$, $0 < c_j < \overline{c}_j < 2$, **(C5)**: $\widetilde{b}_{j,k}, \widetilde{c}_{j,k} \geq 0$ and $\lim_{k\to\infty} \widetilde{b}_{j,k} = \lim_{k\to\infty} \widetilde{c}_{j,k} = 0$, and **(C6)**: $0 < \underline{d} \leq \beta \leq \overline{d} < 1$.

The algorithm is presented as follows.

Algorithm 3.1. Initialization. Given $\beta \in (0,1)$. Let $x^{-1}, x^0, x^1 \in H_1$ be arbitrary, $\{\tau_k\}$, $\{\mu_k\} \subset \mathbb{R}_+$ be bounded sequences. Set $k := 1$.

Step 1. Compute $w^k = (I - \alpha_k A)x^k + \theta_k(x^k - x^{k-1}) + \delta_k(x^{k-1} - x^{k-2})$, where

$$\theta_k = \begin{cases} \min\left\{\frac{\eta_k}{\|x^k - x^{k-1}\|}, \tau_k\right\}, & \text{if } x^k \neq x^{k-1}, \\ \tau_k, & \text{otherwise,} \end{cases} \tag{θ_k}$$

and

$$\delta_k = \begin{cases} \max\left\{\frac{-\eta_k}{\|x^{k-1} - x^{k-2}\|}, -\mu_k\right\}, & \text{if } x^{k-1} \neq x^{k-2}, \\ -\mu_k, & \text{otherwise.} \end{cases} \tag{δ_k}$$

Step 2. Compute $y_j^k = P_{C_j} F_j w^k$.
Step 3. Compute $v_j^k = F_j w^k - y_j^k$.
Step 4. Compute $u_j^k = F_j w^k - \beta_{j,k} v_j^k$, where

$$\beta_{j,k} = \begin{cases} \frac{(\widetilde{b}_{j,k} + b_j)\langle F_j w^k - y_j^k, v_j^k\rangle}{\|v_j^k\|^2}, & \text{if } v_j^k \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{$\beta_{j,k}$}$$

Step 5. Compute $z^k = w^k - \beta\gamma_{1,k} F_1^*(F_1 w^k - u_1^k) - (1-\beta)\gamma_{2,k} F_2^*(F_2 w^k - u_2^k)$, where

$$\gamma_{j,k} = \begin{cases} \frac{(\widetilde{c}_{j,k} + c_j)\|F_j w^k - u_j^k\|^2}{\|F_j^*(F_j w^k - u_j^k)\|^2}, & \text{if } \|F_j^*(F_j w^k - u_j^k)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{$\gamma_{j,k}$}$$

Step 6. Compute $x^{k+1} = (1-\xi_k)w^k + \xi_k z^k$.
Set $k := k+1$ and go to **Step 1**.

Remark 3.1. From Assumption 3.1 and (θ_k), (δ_k), we get $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = 0$ and $\lim_{k\to\infty} \frac{|\delta_k|}{\alpha_k} \times \|x^{k-1} - x^{k-2}\| = 0$. Therefore, there exist $M_1 > 0$, $M_2 > 0$ and a positive integer $k_0 \in \mathbb{N}$ such that $\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq M_1$, $\frac{|\delta_k|}{\alpha_k}\|x^{k-1} - x^{k-2}\| \leq M_2$, $\forall k \geq 1$ and $b_{j,k} + b_j \in (0,2)$, $c_{j,k} + c_j \in (0,2)$, $\forall k \geq k_0$, $j = 1,2$.

Lemma 3.1. Let $\{w^k\}$ and $\{z^k\}$ be sequences generated by Algorithm 3.1 under Assumption 3.1. For every $x \in \Gamma$, $\forall k \geq k_0$, the following inequality holds

$$\begin{aligned} \|z^k - x\|^2 \leq \|w^k - x\|^2 &- \beta\gamma_{1,k}[1 - (c_{1,k} + c_1)]\|F_1 w^k - u_1^k\|^2 \\ &- (1-\beta)\gamma_{2,k}[1 - (c_{2,k} + c_2)]\|F_2 w^k - u_2^k\|^2. \end{aligned} \tag{2}$$

Proof. Since $x \in \Gamma$, $F_j x \in C_j$, $j = 1,2$. From Lemma 2.1 and Step 3, we get

$$\begin{aligned} &\langle y_j^k - F_j x, v_j^k\rangle = \langle y_j^k - F_j x, F_j w^k - y_j^k\rangle \geq 0,\ j = 1,2. \\ \Rightarrow &\langle F_j w^k - F_j x, v_j^k\rangle \geq \langle F_j w^k - y_j^k, v_j^k\rangle,\ j = 1,2. \end{aligned} \tag{3}$$

It follows from Step 4, Remark 3.1 and (3) that

$$\|u_j^k - F_j x\|^2 = \|F_j w^k - F_j x\|^2 + \beta_{j,k}^2\|v_j^k\|^2 - 2\beta_{j,k}\langle F_j w^k - F_j x, v_j^k\rangle$$

$$\leq \|F_j w^k - F_j x\|^2 + \beta_{j,k}^2 \|v_j^k\|^2 - 2\beta_{j,k}\langle F_j w^k - y_j^k, v_j^k\rangle$$
$$\leq \|F_j w^k - F_j x\|^2, \ \forall k \geq k_0, \ j = 1,2. \tag{4}$$

Since the function $\|.\|$ is convex, from Step 5, (4) and the property of adjoint operators, we obtain

$$\|z^k - x\|^2 = \left\|\beta[w^k - \gamma_{1,k}F_1^*(F_1 w^k - u_1^k) - x] + (1-\beta)[w^k - \gamma_{2,k}F_2^*(F_2 w^k - u_2^k) - x]\right\|^2$$
$$\leq \beta\|w^k - \gamma_{1,k}F_1^*(F_1 w^k - u_1^k) - x\|^2 + (1-\beta)\|w^k - \gamma_{2,k}F_2^*(F_2 w^k - u_2^k) - x\|^2$$
$$\leq \|w^k - x\|^2 - \beta\gamma_{1,k}[1-(c_{1,k}+c_1)]\|F_1 w^k - u_1^k\|^2 - (1-\beta)\gamma_{2,k}[1-(c_{2,k}+c_2)]\|F_2 w^k - u_2^k\|^2.$$

Lemma 3.2. Let $\{x^k\}$ be the sequence generated by Algorithm 3.1 under Assumption 3.1, we have that $\{x^k\}$ is a bounded sequence.

Proof. Fixing a point $x \in \Gamma$, from Step 1 and Remark 3.1 we have

$$\|w^k - x\| = \|(I-\alpha_k A)x^k - (I-\alpha_k A)x - \alpha_k Ax + \theta_k(x^k - x^{k-1}) + \delta_k(x^{k-1} - x^{k-2})\|$$
$$\leq \|(I-\alpha_k A)x^k - (I-\alpha_k A)x\| + \alpha_k\|Ax\| + \alpha_k M_3, \ M_3 := M_1 + M_2. \tag{5}$$

From $\lim_{k\to\infty}\alpha_k = 0$, for any $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1, \ \alpha_k < \varepsilon$. Let $k_* = \max\{k_0, k_1\}$, we have $\alpha_k < \varepsilon, \ \forall k \geq k_*$. It follows from the convexity of the norm, the η-strongly monotonicity end the ℓ-Lipschitz continuity of A, for all $k \geq k_*$ we get

$$\|(I-\alpha_k A)x^k - (I-\alpha_k A)x\| \leq \left(1 - \frac{\alpha_k}{\varepsilon}\right)\|x^k - x\| + \frac{\alpha_k}{\varepsilon}\|(x^k - x) - \varepsilon(Ax^k - Ax)\|$$
$$\leq \left(1 - \frac{\alpha_k}{\varepsilon}\right)\|x^k - x\| + \frac{\alpha_k}{\varepsilon}\sqrt{1-\varepsilon(2\eta - \varepsilon\ell^2)}\|x^k - x\|$$
$$= \left(1 - \frac{\rho\alpha_k}{\varepsilon}\right)\|x^k - x\|, \ \forall k \geq k_*, \tag{6}$$

where $\rho = 1 - \sqrt{1-\varepsilon(2\eta - \varepsilon\ell^2)} \in (0,1)$ and $\varepsilon \in \left(0, \frac{2\eta}{\ell^2}\right)$. From (5) and (6), we get

$$\|w^k - x\| \leq \left(1 - \frac{\rho\alpha_k}{\varepsilon}\right)\|x^k - x\| + \alpha_k\|Ax\| + \alpha_k M_3, \ \forall k \geq k_*. \tag{7}$$

It follows from Step 6, Remark 3.1, (2) and the convexity of the norm that

$$\|x^{k+1} - x\| \leq (1-\xi_k)\|w^k - x\| + \xi_k\|z^k - x\| \leq \|w^k - x\|$$
$$\leq \left(1 - \frac{\alpha_k}{\varepsilon}\rho\right)\|x^k - x\| + \frac{\alpha_k}{\varepsilon}\rho.\frac{\varepsilon(\|Ax\| + M_3)}{\rho} \leq \max\left\{\|x^k - x\|, \frac{\varepsilon(\|Ax\| + M_3)}{\rho}\right\}, \ \forall k \geq k_*$$
$$\Rightarrow \|x^{k+1} - x\| \leq \max\left\{\|x^{k_*} - x\|, \frac{\varepsilon(\|Ax\| + M_3)}{\rho}\right\}, \ \forall k \geq k_*.$$

This implies that the sequence $\{x^k\}$ is bounded.

Lemma 3.3. Let $\{w^k\}$ and $\{z^k\}$ be two sequences generated by Algorithm 3.1 under Assumption 3.1 with subsequences $\{w^{k_l}\}$ and $\{z^{k_l}\}$, respectively. If $\{w^{k_l}\}$ converges strongly to a point $\hat{x} \in H_1$ and $\lim_{l\to\infty}\|w^{k_l} - z^{k_l}\| = 0$, then $\hat{x} \in \Gamma$.

Proof. Since $\lim_{l\to\infty}\|w^{k_l} - z^{k_l}\| = 0$, it follows from (2) that

$$\beta\gamma_{1,k_l}[1-(c_{1,k_l}+c_1)]\|F_1 w^{k_l} - u_1^{k_l}\|^2 + (1-\beta)\gamma_{2,k_l}[1-(c_{2,k_l}+c_2)]\|F_2 w^{k_l} - u_2^{k_l}\|^2$$

$$\leq \|w^{k_l}-x\|^2-\|z^{k_l}-x\|^2 \leq \|w^{k_l}-z^{k_l}\|^2+2\|w^{k_l}-z^{k_l}\|\|z^{k_l}-x\|$$

tend to 0 as $l \to \infty$. Therefore,

$$\begin{aligned}&\|F_jw^{k_l}-u_j^{k_l}\| \to 0 \text{ as } l \to \infty,\ j=1,2\\ &\|F_j^*(F_jw^{k_l}-u_j^{k_l})\| \leq \|F_j\|\|F_jw^{k_l}-u_j^{k_l}\| \to 0 \text{ as } l \to \infty,\ j=1,2.\end{aligned} \tag{8}$$

From Step 3, Step 4 and Remark 3.1, we get

$$\begin{aligned}&\|F_jw^{k_l}-y_j^{k_l}\|^2=\langle F_jw^{k_l}-y_j^{k_l},v_j^{k_l}\rangle=\frac{1}{(b_{j,k_l}+b_j)}\|F_jw^{k_l}-y_j^{k_l}\|\|F_jw^{k_l}-u_j^{k_l}\|\\ \Rightarrow\ &\|F_jw^{k_l}-y_j^{k_l}\|=\frac{1}{(b_{j,k_l}+b_j)}\|F_jw^{k_l}-u_j^{k_l}\| \to 0 \text{ as } l\to\infty,\ j=1,2.\end{aligned} \tag{9}$$

Since $w^{k_l} \to \hat{x}$ and F_j is a bounded linear operator for each $j=1,2$, we have $F_jw^{k_l} \to F_j\hat{x}$. Thus, we get $y_j^{k_l} \to F_j\hat{x},\ j=1,2$. Therefore, we obtain $F_j\hat{x} \in C_j$. Thus $\hat{x} \in \Gamma$.

Proposition 3.1. Let $\{x^k\}$ be the sequence generated by Algorithm 3.1 under Assumption 3.1. Then, for any $x \in \Gamma$, the following inequality holds

$$\|x^{k+1}-x\|^2 \leq (1-\alpha_k\tau)\|x^k-x\|^2+\alpha_k\tau d_k,\ k \geq k_*, \tag{10}$$

where $k_* \in \mathbb{N},\ \tau=2\eta-\varepsilon\ell^2 \in (0,1)$ and $\{d_k\}$ is a sequence of real numbers.

Proof. Let $x \in \Gamma$. From Step 1, (5), (6) and the convexity of the $\|.\|^2$, we get

$$\|w^k-x\|^2 \leq (1-\alpha_k\tau)\|x^k-x\|^2+\alpha_k\tau d_k,\ k \geq k_*, \tag{11}$$

where $\tau=2\eta-\varepsilon\ell^2 \in (0,1)$ and

$$\begin{aligned}d_k := \frac{1}{\tau}\Big[&2\langle x-x^k,Ax\rangle+2\alpha_k\|Ax^k\|\|Ax\|+2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|[\|x^k-x\|+\alpha_k\|Ax\|]\\ &+2\frac{|\delta_k|}{\alpha_k}\|x^{k-1}-x^{k-2}\|[\|x^k-x\|+\alpha_k\|Ax\|]+\frac{\theta_k^2}{\alpha_k}\|x^k-x^{k-1}\|^2\\ &+2\frac{\theta_k|\delta_k|}{\alpha_k}\|x^k-x^{k-1}\|\|x^{k-1}-x^{k-2}\|+\frac{\delta_k^2}{\alpha_k}\|x^{k-1}-x^{k-2}\|^2\Big],\ k \geq k_*.\end{aligned} \tag{12}$$

From Step 6, (2) and (11), we obtain

$$\begin{aligned}\|x^{k+1}-x\|^2 &\leq (1-\xi_k)\|w^k-x\|^2+\xi_k\|z^k-x\|^2-\xi_k(1-\xi_k)\|w^k-z^k\|^2\\ &=\|w^k-x\|^2-\xi_k(1-\xi_k)\|w^k-z^k\|^2-\xi_k\sum_{j=1}^{2}\delta_{j,k}\gamma_{j,k}[1-(c_{j,k}+c_j)]\|F_jw^k-u_j^k\|^2\\ &\leq (1-\alpha_k\tau)\|x^k-x\|^2+\alpha_k\tau d_k,\ k \geq k_*.\end{aligned} \tag{13}$$

Theorem 3.1. *Let $\{x^k\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then $\{x^k\}$ converges strongly to the unique solution of the* VIP(A,Γ).

Proof. Since $\Gamma \neq \varnothing$, the VIP(A,Γ) has a unique solution, denoted by x^*. In particular, $x^* \in \Gamma$, i.e., $F_jx^* \in C_j, \forall j=1,2$. Then, from Proposition 3.1 we obtain

$$\|x^{k+1}-x^*\|^2 \leq (1-\alpha_k\tau)\|x^k-x^*\|^2+\alpha\tau d_k^*, \tag{14}$$

where

$$d_k^* := \frac{1}{\tau}\Bigg[2\langle x^* - x^k, Ax^*\rangle + 2\alpha_k\|Ax^k\|\|Ax^*\| + 2\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\|[\|x^k - x^*\| + \alpha_k\|Ax^*\|]$$
$$+ 2\frac{|\delta_k|}{\alpha_k}\|x^{k-1} - x^{k-2}\|[\|x^k - x^*\| + \alpha_k\|Ax^*\|] + \frac{\theta_k^2}{\alpha_k}\|x^k - x^{k-1}\|^2$$
$$+ 2\frac{\theta_k|\delta_k|}{\alpha_k}\|x^k - x^{k-1}\|\|x^{k-1} - x^{k-2}\| + \frac{\delta_k^2}{\alpha_k}\|x^{k-1} - x^{k-2}\|^2\Bigg],\ k \geq k_*. \quad (15)$$

Suppose that $\{\|x^{k_l} - x^*\|\} \subseteq \{\|x^k - x^*\|\}$ such that $\liminf_{l\to\infty}(\|x^{k_l+1} - x^*\| - \|x^{k_l} - x^*\|) \geq 0$. From (11), (13) and $\lim_{l\to\infty}\alpha_{k_l} = 0$, we obtain

$$\xi_{k_l}(1-\xi_{k_l})\|w^{k_l} - z^{k_l}\|^2 + \xi_{k_l}\sum_{j=1}^{2}\delta_{j,k_l}\gamma_{j,k_l}[1-(c_{j,k_l}+c_j)]\|F_jw^{k_l} - u_j^{k_l}\|^2$$
$$\leq (\|x^{k_l} - x^*\|^2 - \|x^{k_l+1} - x^*\|^2) - \alpha_{k_l}\tau\|x^{k_l} - x^*\|^2 + \alpha_{k_l}\tau d_{k_l}^*,\ k \geq k_*$$

tend to 0 as $l \to \infty$. Thus

$$\|w^{k_l} - z^{k_l}\|^2 \to 0 \text{ and } \|F_jw^{k_l} - u_j^{k_l}\|^2 \to 0,\ \forall j = 1,2. \quad (16)$$

From Step 1, Step 6 and by Remark 3.1, we have, as $l \to \infty$,

$$\|w^{k_l} - x^{k_l}\| \leq \alpha_{k_l}\|Ax^{k_l}\| + \theta_{k_l}\|x^{k_l} - x^{k_l-1}\| + |\delta_k|\|x^{k_l-1} - x^{k_l-2}\| \to 0$$
$$\Rightarrow \|z^{k_l} - x^{k_l}\| \leq \|z^{k_l} - w^{k_l}\| + \|w^{k_l} - x^{k_l}\| \to 0$$
$$\Rightarrow \|x^{k_l+1} - x^{k_l}\| \leq (1-\xi_{k_l})\|w^{k_l} - x^{k_l}\| + \xi_{k_l}\|z^{k_l} - x^{k_l}\| \to 0$$
$$\Rightarrow \|x^{k_l+1} - w^{k_l}\| \leq \|x^{k_l+1} - x^{k_l}\| + \|x^{k_l} - w^{k_l}\| \to 0. \quad (17)$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ of $\{x^k\}$ such that $x^{k_l} \to \hat{x}$, we get $w^{k_l} \to \hat{x}$. Thus, we obtain from Lemma 3.3 that $\hat{x} \in \Gamma$.
By the boundedness of $\{x^{k_l}\}$, there exists a subsequence $\{x^{k_{l_m}}\}$ of $\{x^{k_l}\}$ that converges strongly $\widetilde{x} \in H_1$ such that $\limsup_{l\to\infty}\langle Ax^*, x^* - x^{k_l}\rangle = \lim_{m\to\infty}\langle Ax^*, x^* - x^{k_{l_m}}\rangle = \langle Ax^*, x^* - \widetilde{x}\rangle$. Also, we obtain from Lemma 3.3 that $\widetilde{x} \in \Gamma$. Since x^* is the solution to the VIP(A,Γ), we have $\langle Ax^*, \widetilde{x} - x^*\rangle \geq 0$. This implies $\limsup_{l\to\infty}\langle Ax^*, x^* - x^{k_l}\rangle \leq 0$. By Remark 3.1, the boundedness of $\{x^{k_l}\}$ and $\{Ax^{k_l}\}$ we have $\limsup_{l\to\infty} d_{k_l}^* \leq 0$. Consequently, by invoking Lemma 2.5 it follows from (10) that $\{\|x^k - x^*\|\}$ converges strongly to 0 as required.

3.2. Application to image restoration problems

In this section, we present an application for image restoration problems using our main result. We provide some comparisons to the CQ algorithm of Byrne [5] and the inertial iterative algorithm of Nguyen et al. [9].

For a gray scale image of M pixels width and N pixels height, each pixel value ranges from 0 to 255. Let $D = M \times N$, then the real space $\mathbb{R}^D$ is equipped with the Euclidean norm and $C = [0,255]^D$. To estimate the approximate value of the vector x, representing the original image, we consider the convex minimization model $\min_{x\in C}\|Fx - y\|$ where y is the observed blurred image and F is the blurring matrix. Choose $Q = \{y\}$, then this model becomes the (SFP). Therefore, we can apply our algorithm to solve the image restoration problem.

To determine the effectiveness of the algorithm, we need a measure of the image quality of the restored images. We determine the peak signal-to-noise ratio (SNR) in decibels (dB) as follows

$$\text{SNR } = 20\log_{10}\frac{\|x^*\|}{\|x-x^*\|}, \tag{18}$$

where x^* is the original image and x is the restored image. It can be seen that a higher SNR value indicates a better-restored image.

The test images are the cameraman and coin images from MATLAB built-in demo images in Figure 1 for the test case of a grayscale image with a Gaussian blur size of $[9\times 9]$ and a standard deviation of 4 in Figure 2.

Figure 1. *Original images of a cameraman (*256×256 *pixels) and coins (*246×300 *pixels)*

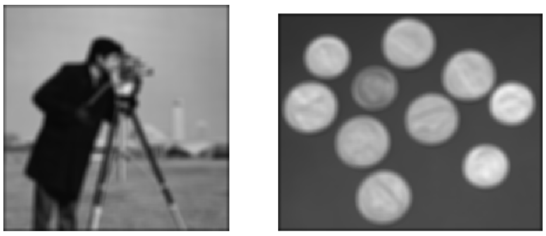

Figure 2. *Images degraded by a Gaussian blur of size* 9×9 *and standard deviation* 4

We aim to compare our Algorithm 3.1 to CQ algorithm of Byrne [5] and inertial iterative algorithm of Nguyen et al. [9].To begin, set the initial point $x^{-1} = x^0 = x^1 = (0,\ldots,0) \in \mathbb{R}^D$. In our test, we select the parameters as follows

- In our algorithm, we choose $A(x) = 0.015x$, $\alpha_k = \frac{1}{k+1}$, $\mu_k = \frac{1}{(k+1)^2}$, $\tau_k = \frac{0.99k}{k+0.001}$, $\eta_k = \frac{1}{(k+1)^{1.01}}$, $\beta = 0.25$, $b_1 = b_2 = 1.55$, $c_1 = c_2 = 1.55$, $b_{1,k} = b_{2,k} = \frac{1}{(3k+1)^3}$, $c_{1,k} = c_{2,k} = \frac{1}{(5k+1)^5}$, $\xi_k = \frac{k}{k+1}$, $\forall k \geq 1$.
- In Algorithm of Byrne, we choose $\lambda = \frac{1}{L} \in (0, \frac{2}{L})$ where L being the largest eigenvalue of F^*F with the blur matrix F.
- In Algorithm of Nguyen et al., we choose $\eta = 0.5,\ \theta = 0.25,\ f(x) = \tau x = 0.985x$ and $\rho_k = 1$, $\eta_k = \frac{1}{(k+1)^{1.01}},\ e_k = 10^{-8},\ \alpha_k = \frac{1}{k+1}, \forall k \geq 1$.

The results are presented in Table 1 and Figure 3, we observe that our Algorithm has higher SNR values. It shows that our algorithm has a better convergence behavior than the algorithm of Byrne and the algorithm of Nguyen et al. for each case. Next, in Figure 4, we demonstrate the recovered images by using Algorithms for the Gaussian blur of size 9×9 and the number of iterations is 1500. From Figures 3, 4 and Tables 1, it shows the applicability and effciency of the proposed method for solving the image restoration problems which is the application of the (SFP).

Table 1. *Numerical comparison for Algorithms of cameraman image and coins image*

		SNR(dB)		
	Iter. (k)	**Our Alg.**	**Byrne's Alg.**	**Nguyen et al.'s Alg.**
Cameraman image	500	24.0620	20.6973	21.8555
	1000	24.5660	21.4651	22.4774
	1500	24.6434	21.9191	22.9092
Coins image	500	24.9113	22.5663	23.2941
	1000	25.1662	23.2684	24.0323
	1500	25.6406	23.6996	24.5049

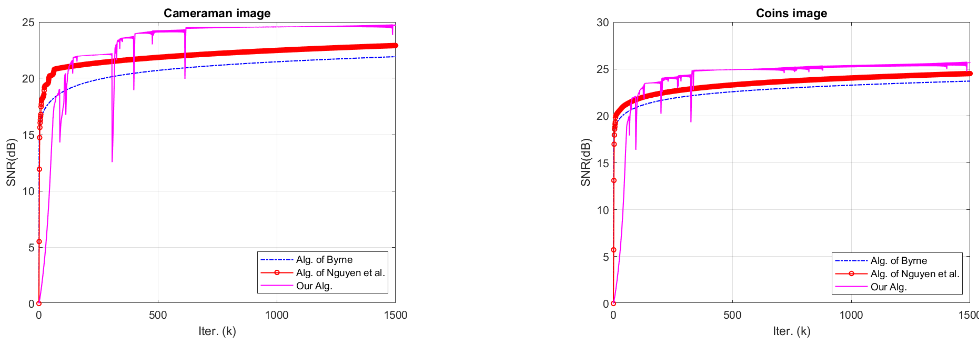

Figure 3. *Graphs of SNR for Algorithms of the cameraman image and coins image*

Our Alg. Byrne's Alg. Nguyen et al.'s Alg.

Figure 4. *Recovered images with the Gaussian blur of size* 9×9

4. Conclusion

In this paper, we have described an algorithm for solving the (SFP) by combining the inertial method with the relaxation method and self-adaptive step size. By effectively incorporating convex combinations into the algorithm, we have demonstrated the strong convergence results of the proposed method under the given assumptions. The combination of inertial and relaxation methods has improved the convergence speed of our algorithm, outperforming the compared methods in a practical application experiment for image restoration.

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