# MULTI-STEP INERTIAL PROXIMAL POINT ALGORITHM FOR THE SPLIT COMMON SOLUTION PROBLEM OF MONOTONE OPERATOR EQUATIONS

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**ABSTRACT**

The split feasibility problem plays an important role in various fields such as optimization theory, signal processing, image reconstruction, game theory, and several other areas. Based on the proximal point method combined with a multi-step inertial term, the author proposes a new algorithm to solve a generalized form of the split feasibility problem. Specifically, this paper introduces a multi-step inertial proximal point algorithm to solve the split common solution problem for monotone operator equations with multiple out- put sets in real Hilbert spaces. Under suitable conditions on the control parameters, the sequence generated by the proposed algorithm converges weakly to a solution of the problem. Additionally, the paper presents the applicability of the proposed algorithm to related problems such as the split common fixed point problem, the split feasibility problem, and the split common null point problem with multiple output sets, thereby demonstrating the effectiveness and potential extensibility of the proposed algorithm.

# THUẬT TOÁN ĐIỂM GẦN KỀ ĐA QUÁN TÍNH CHO BÀI TOÁN NGHIỆM CHUNG TÁCH CỦA LỚP PHƯƠNG TRÌNH TOÁN TỬ ĐƠN ĐIỆU

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**TÓM TẮT**

Bài toán chấp nhận tách đóng vai trò quan trọng trong nhiều lĩnh vực như lý thuyết tối ưu, xử lý tín hiệu, khôi phục ảnh, lý thuyết trò chơi và một số lĩnh vực khác. Dựa trên phương pháp điểm gần kề kết hợp với thành phần quán tính nhiều bước tác giả đề xuất một thuật toán mới để giải một dạng bài toán tổng quát của lớp bài toán chấp nhận tách. Cụ thể, bài báo này đề xuất thuật toán điểm gần kề đa quán tính để giải bài toán nghiệm chung tách của lớp phương trình toán tử đơn điệu với đa tập đầu ra trong không gian Hilbert thực. Dưới các điều kiện thích hợp đối với các tham số điều khiển thì dãy lặp sinh ra từ thuật toán đề xuất hội tụ yếu tới nghiệm của bài toán. Ngoài ra, bài báo còn trình bày khả năng ứng dụng thuật toán trong các bài toán liên quan như điểm bất động chung tách, chấp nhận tách, và không điểm chung tách với đa tập đầu ra, qua đó khẳng định tính hiệu quả và tiềm năng mở rộng của thuật toán đề xuất.

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## 1. Introduction

In this paper, we study the following generalized split feasibility model, referred to as the *split common solution problem with multiple output sets for monotone operator equations* (SCSP-MOS, for short). Let $\mathcal{X}$ and $\mathcal{Y}_i,\ (i = 1, 2, \ldots, N)$ be real Hilbert spaces.

Let $F :\ \mathcal{X} \to \mathcal{X}$ be a $\gamma$-inverse strongly monotone operator, and $F_i :\ \mathcal{Y}_i \to \mathcal{Y}_i$ be $\gamma_i$-inverse strongly monotone operators $(i = 1, 2, \ldots, N)$ . Let $M_i :\ \mathcal{X} \to \mathcal{Y}_i\ (i = 1, 2, \ldots, N)$ be bounded linear operators. Let $h \in \mathcal{X}$ and $h_i \in \mathcal{Y}_i\ (i = 1, 2, \ldots, N)$ be given points. We denote

$$\begin{aligned}
&\mathcal{S}_0 = \{x \in \mathcal{X} : F(x) = h\},\\
&\mathcal{S}_i = \{x \in \mathcal{X} : F_i(M_i x) = h_i\}, \quad i = 1, 2, \ldots, N,
\end{aligned}$$

and suppose that

$$\mathcal{S} := \mathcal{S}_0 \cap \left( \bigcap_{i=1}^{N} \mathcal{S}_i \right) \neq \emptyset.$$

Then, the SCSP-MOS is to find an element $x \in \mathcal{S}$.

The SCSP-MOS serves as a unified framework that encompasses various split problems, such as the *split common fixed point problem with multiple output sets*, the *split feasibility problem with multiple output sets* and the *split common null point problem with multiple output sets*. These problems are generalized forms of the split feasibility problem, thus offering a wider range of applications. For more details, see references [1]-[5], as well as other related literature.

The proximal point algorithm was first introduced by Martinet [6] and later extended by Alvarez and Attouch [7] with inertial terms. Zhang et al.[8] proposed the multi-step inertial proximal contraction algorithm for monotone variational inclusion problems. Numerical results show that this method outperforms the classical inertial proximal point algorithm.

In recent years, Ha et al. [9] and Reich et al. [10] have introduced inertial proximal point methods for solving the SCSP-MOS.

From the above facts, in this paper, we introduce a new algorithms for solving the SCSP-MOS using the multi-step inertial proximal algorithm.

## 2. Premiliars

Let $\mathcal{X}$ be a Hilbert space. We denote by $\langle \cdot, \cdot \rangle$, $\| \cdot \|$, $\to$, and $\rightharpoonup$ the inner product, the norm, the strong convergence, and the weak convergence in $\mathcal{X}$, respectively.

**Definition 2.1.** Let $F : \mathcal{X} \to \mathcal{X}$ be an operator. We call

(i) $F$ is monotone if $\langle x - y, F(x) - F(y) \rangle \geq 0, \quad \forall x, y \in \mathcal{X}$.

(ii) $F$ is $\alpha$-inverse strongly monotone if there exists a positive constant $\alpha > 0$ such that

$$\langle x - y, F(x) - F(y) \rangle \geq \alpha \| F(x) - F(y) \|^2, \quad \forall x, y \in \mathcal{X}.$$

The following lemma is used in the proof of the main results of this paper.

**Lemma 2.2.** [8] Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences, and let $c = (c_i)_m \in \mathbb{R}^m$ such that $a_{n+1} \leq \sum_{i=0}^{m-1} c_i a_{n-i} + b_n$, for all $n \geq m$. Assume further that

$$0 \leq \sum_{i=0}^{m-1} c_i < 1 \quad \text{and} \quad \sum_{n=0}^{\infty} b_n < \infty.$$

Then the sequence $\{a_k\}$ is summable, that is $\sum_{k=0}^{\infty} a_k < \infty$.

## 3. Main results

In this section, we denote $G(x) := F(x) + \sum_{i=1}^{N} M_i^* F_i(M_i x)$. We begin with the following observations.

**Remark 3.1.** By applying Lemma 2.3 in [10], we deduce that the operator $I + G$ is a bijection. Hence, we can define a mapping $J^G : \mathcal{X} \to \mathcal{X}$, by

$$J^G(x) = (I + G)^{-1}(x), \text{for all } x \in \mathcal{X}.$$

In oder to find a solution of the SCSP-MOS, we introduce the following algorithm.

**Algorithm 3.2. Step 1.** Choose real sequences $\{\alpha_{j,n}\} \subset (-1, 1)$ for all $j = 1, 2, \ldots, m$. Select arbitrary points $x_{-m}, x_{-(m-1)}, x_{-(m-2)}, \ldots, x_0 \in \mathcal{X}$ and set $n := 0$.
**Step 2.** Compute

$$y_n = x_n + \sum_{j=1}^{m} \alpha_{j,n}(x_{n+1-j} - x_{n-j}). \tag{1}$$

**Step 3.** Compute $x_{n+1}$ as follows

$$x_{n+1} = J^G(y_n + h + \sum_{i=1}^{N} M_i^* h_i). \tag{2}$$

**Step 4.** Set $n \leftarrow n + 1$, and go to Step 2.

The following theorem confirms the convergence of Algorithm 3.2.

**Theorem 3.3.** *Suppose that the sequence $\{\alpha_{j,n}\}$ for all $j = 1, 2, \ldots, m$ is chosen to satisfy the following conditions:*

*(C1)* $\sum\limits_{j=1}^{m} \overline{\alpha}_j < 1$, *where* $\overline{\alpha}_j = \sup_{n\in\mathbb{N}} |\alpha_{j,n}|$, *for all* $j = 1, 2, \ldots, m$.

*(C2)* $\sum\limits_{n=1}^{\infty} \max_{j=\overline{1,m}} |\alpha_{j,n}| \sum\limits_{j=1}^{m} \|x_{n+1-j} - x_{n-j}\|^2 < \infty.$

*Then the sequence $\{x_n\}$ converges weakly to a solution of the SCSP-MOS.*

*Proof.* Using Remark 3.1, we see that the sequence $\{x_n\}$ is well defined. Let $s \in S$, that is, $F(s) = h$ and $F_i(M_i s) = h_i$, for all $i = 1, 2, \ldots, N$. From (1), we have

$$\begin{aligned}\|y_n - s\|^2 &= \|x_n + \sum_{j=1}^{m} \alpha_{j,n}(x_{n+1-j} - x_{n-j}) - s\|^2 \\ &= \|x_n - s\|^2 + \|\sum_{j=1}^{m} \alpha_{j,n}(x_{n+1-j} - x_{n-j})\|^2 + 2\sum_{j=1}^{m} \alpha_{j,n}\langle x_n - s, x_{n+1-j} - x_{n-j}\rangle, \end{aligned} \tag{3}$$

and

$$\begin{aligned}\|x_{n+1} - y_n\|^2 &= \|x_{n+1} - x_n - \sum_{j=1}^{m} \alpha_{j,n}(x_{n+1-j} - x_{n-j})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \|\sum_{j=1}^{m} \alpha_{j,n}(x_{n+1-j} - x_{n-j})\|^2 - 2\sum_{j=1}^{m} \alpha_{j,n}\langle x_{n+1} - x_n, x_{n+1-j} - x_{n-j}\rangle. \end{aligned} \tag{4}$$

Rewrite (2) in the following form

$$F(x_{n+1}) + \sum_{i=1}^{N} M_i^* F_i(M_i x_{n+1}) - h - \sum_{i=1}^{N} M_i^* h_i + x_{n+1} = y_n. \tag{5}$$

From (5), we have

$$\begin{aligned}\langle x_{n+1} - s, y_n - s\rangle &= \langle x_{n+1} - s, F(x_{n+1}) + \sum_{i=1}^{N} M_i^* F_i(M_i x_{n+1}) - h - \sum_{i=1}^{N} M_i^* h_i + x_{n+1} - s\rangle \\ &= \langle x_{n+1} - s, F(x_{n+1}) - h\rangle + \sum_{i=1}^{N}\langle x_{n+1} - s, M_i^* F_i(M_i x_{n+1}) - M_i^* h_i\rangle \\ &\quad + \langle x_{n+1} - s, x_{n+1} - s\rangle \\ &= \langle x_{n+1} - s, F(x_{n+1}) - F(s)\rangle + \sum_{i=1}^{N}\langle M_i x_{n+1} - M_i s, F_i(M_i x_{n+1}) - F_i(M_i s)\rangle \\ &\quad + \|x_{n+1} - s\|^2. \end{aligned} \tag{6}$$

Since $F$ is a $\gamma$-inverse strongly monotone operator, $F_i$ $(i = 1, 2, \dots, N)$ are $\gamma_i$-inverse strongly monotone operators, respectively, and from (6), we obtain

$$\begin{aligned}&\gamma\|F(x_{n+1}) - F(s)\|^2 + \sum_{i=1}^{N} \gamma_i \|F_i(M_i x_{n+1}) - F_i(M_i s)\|^2 + \|x_{n+1} - s\|^2 \\ &\le \frac{1}{2}\left(\|x_{n+1} - s\|^2 + \|y_n - s\|^2 - \|x_{n+1} - y_n\|^2\right). \end{aligned} \tag{7}$$

We can rewrite (7) in the following form:

$$\|x_{n+1} - s\|^2 \le \|y_n - s\|^2 - \|x_{n+1} - y_n\|^2 - 2R_{n+1}(s), \tag{8}$$

where

$$R_{n+1}(s) = \gamma\|F(x_{n+1}) - F(s)\|^2 + \sum_{i=1}^{N} \gamma_i \|F_i(M_i x_{n+1}) - F_i(M_i s)\|^2.$$

Using (3) and (4) in (8), we have

$$\begin{aligned}\|x_{n+1} - s\|^2 &\le \|x_n - s\|^2 - \|x_{n+1} - x_n\|^2 - 2R_{n+1}(s) \\ &\quad + 2\sum_{j=1}^{m} \alpha_{j,n}\langle x_{n+1} - s, x_{n+1-j} - x_{n-j}\rangle. \end{aligned} \tag{9}$$

Beside, we have

$$\begin{aligned}&\langle x_{n+1} - s, x_{n+1-j} - x_{n-j}\rangle \\ &= \langle x_{n+1} - x_n, x_{n+1-j} - x_{n-j}\rangle + \langle x_n - x_{n+1-j}, x_{n+1-j} - x_{n-j}\rangle \\ &\quad + \langle x_{n+1-j} - s, x_{n+1-j} - x_{n-j}\rangle \\ &= \langle x_{n+1} - x_n, x_{n+1-j} - x_{n-j}\rangle + \frac{1}{2}\left(\|x_n - x_{n-j}\|^2 - \|x_n - x_{n+1-j}\|^2\right) \\ &\quad + \frac{1}{2}\left(\|x_{n+1-j} - s\|^2 - \|x_{n-j} - s\|^2\right). \end{aligned}$$

From (9), we get

$$\begin{aligned}\|x_{n+1}-s\|^2 \leq \|x_n-s\|^2 &- \|x_{n+1}-x_n\|^2 - 2R_{n+1}(s)\\ &+ 2\sum_{j=1}^{m}\alpha_{j,n}\langle x_{n+1}-x_n, x_{n+1-j}-x_{n-j}\rangle\\ &+ \sum_{j=1}^{m}\alpha_{j,n}\left(\|x_n-x_{n-j}\|^2 - \|x_n-x_{n+1-j}\|^2\right)\\ &+ \sum_{j=1}^{m}\alpha_{j,n}\left(\|x_{n+1-j}-s\|^2 - \|x_{n-j}-s\|^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-s\|^2 - \|x_n-s\|^2 \leq &\sum_{j=1}^{m}\alpha_{j,n}\left(\|x_{n+1-j}-s\|^2 - \|x_{n-j}-s\|^2\right)\\ &- \|x_{n+1}-x_n-\sum_{j=1}^{m}\alpha_{j,n}(x_{n+1-j}-x_{n-j})\|^2\\ &+ \sum_{j=1}^{m}|\alpha_{j,n}|\left(\|x_n-x_{n-j}\|^2 + \|x_n-x_{n+1-j}\|^2\right)\\ &+ \|\sum_{j=1}^{m}\alpha_{j,n}(x_{n+1-j}-x_{n-j})\|^2 - 2R_{n+1}(s). \qquad (10)\end{aligned}$$

Set

$$\begin{aligned}&a_n := \|x_n-s\|^2 - \|x_{n-1}-s\|^2,\\ &\omega_n := x_{n+1}-x_n-\sum_{j=1}^{m}\alpha_{j,n}(x_{n+1-j}-x_{n-j}),\\ &b_n := \sum_{j=1}^{m}|\alpha_{j,n}|\left(\|x_n-x_{n-j}\|^2 + \|x_n-x_{n+1-j}\|^2\right) + \|\sum_{j=1}^{m}\alpha_{j,n}(x_{n+1-j}-x_{n-j})\|^2.\end{aligned}$$

It takes from (10), we get

$$a_{n+1} \leq -\|\omega_n\|^2 + \sum_{j=1}^{m}\alpha_{j,n}a_{n-j} + b_n - 2R_{n+1}(s). \qquad (11)$$

This implies the following inequality

$$[a_{n+1}]_+ \leq \sum_{j=1}^{m}\overline{\alpha}_j[a_{n-j}]_+ + b_n,$$

where

$$[\theta]_+ := \max\{\theta, 0\},$$

$$\overline{\alpha}_j = \sup_{n\in\mathbb{N}}|\alpha_{j,n}| \text{ for all } j = 1, 2, \ldots, m.$$

We now proceed to show that the sequence $b_n$ is summable. Indeed, by applying the Cauchy–Schwarz inequality, for any $j \leq m$, we get

$$\begin{aligned} \|x_n - x_{n-j}\|^2 &\leq \left( \sum_{k=1}^{j} \|x_{n+1-k} - x_{n-k}\| \right)^2 \\ &\leq j \sum_{k=1}^{j} \|x_{n+1-k} - x_{n-k}\|^2 \leq m \sum_{j=1}^{m} \|x_{n+1-j} - x_{n-j}\|^2 . \end{aligned} \tag{12}$$

Using the definition of $b_n$, the Cauchy–Schwarz inequality and (12) we obtain

$$\begin{aligned} b_n &\leq \left( \sum_{j=1}^{m} |\alpha_{j,n}| \, \|x_{n+1-j} - x_{n-j}\| \right)^2 + \sum_{j=1}^{m} |\alpha_{j,n}| \left( \|x_n - x_{n-j}\|^2 + \|x_n - x_{n+1-j}\|^2 \right) \\ &\leq \sum_{j=1}^{m} |\alpha_{j,n}| \sum_{j=1}^{m} \|x_{n+1-j} - x_{n-j}\|^2 + 2m \sum_{j=1}^{m} |\alpha_{j,n}| \sum_{j=1}^{m} \|x_{n+1-j} - x_{n-j}\|^2 \\ &\leq (m + 2m^2) \max_j |\alpha_{j,n}| \sum_{j=1}^{m} \|x_{n+1-j} - x_{n-j}\|^2 . \end{aligned}$$

It follows from condition $(C_2)$ that $\sum_{n=1}^{\infty} b_n < \infty$. Combining the above conclusion with condition $(C_1)$, we see that the assumptions of Lemma 2.2 are fulfilled. Hence $\sum_{n=1}^{\infty} [a_n]_+ < \infty$.
Let $z_n := \|x_n - s\|^2 - \sum_{k=1}^{n} [a_k]_+$. Since $\sum_{n=1}^{\infty} [a_n]_+ < \infty$, we see that the sequence $\{z_n\}$ is bouned from below. Moreover, using the definitions of $a_n$ and $[\theta]_+$, it follows that

$$z_{n+1} = \|x_{n+1} - s\|^2 - [a_{n+1}]_+ + \sum_{k=1}^{n} [a_k]_+ \leq \|x_{n+1} - s\|^2 - a_{n+1} + \sum_{k=1}^{n} [a_k]_+ = z_n.$$

Hence, $\{z_n\}$ is a decreasing sequence. We deduce that the sequence $\{z_n\}$ is convergent and so is $\{\|x_n - s\|\}$. Thus, $\lim_{n\to\infty} a_n = 0$. From (11), we have

$$\|\omega_n\|^2 + 2R_{n+1}(s) \leq -a_{n+1} + \sum_{j=1}^{m} \alpha_{j,n} a_{n-j} + b_n.$$

Since $\{\alpha_{j,n}\}$ is bounded, $\lim_{n\to\infty} a_n = 0$, and the sequence $\{b_n\}$ is summable, we get

$$\lim_{n\to\infty} \left[ \|\omega_n\|^2 + 2R_{n+1}(s) \right] = 0.$$

This implies that

$$\|F(x_{n+1}) - h\| \to 0, \tag{13}$$
$$\|F_i(M_i x_{n+1}) - h_i\| \to 0, \tag{14}$$
$$\|\omega_n\| \to 0.$$

So the sequence $\{\|x_n - s\|\}$ is convergent that the sequence $\{x_n\}$ is bounded. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to $s_* \in \mathcal{X}$. We observe that

$$\begin{aligned} 0 &\leq \gamma \|F(x_{n_k}) - F(s_*)\|^2 \leq \langle F(x_{n_k}) - F(s_*), x_{n_k} - s_* \rangle \\ &= \langle F(x_{n_k}) - h, x_{n_k} - s_* \rangle + \langle h - F(s_*), x_{n_k} - s_* \rangle \end{aligned}$$

$$\leq K\|F(x_{n_k}) - h\| + \langle h - F(s_*), x_{n_k} - s_*\rangle, \quad (15)$$

where $K = \sup_k \|x_{n_k} - s_*\|$.
In (15), letting $k \to \infty$ and using $x_{n_k} \rightharpoonup s_*$, we obtain $\|F(x_{n_k}) - F(s_*)\| \to 0$. This, combined with (13), gives $F(s_*) = h$. It means that $s_* \in S_0$.

Since each $M_i$ is a bounded linear operator, the weak convergence $x_{n_k} \rightharpoonup s_*$ implies that $M_i x_{n_k} \rightharpoonup M_i s_*$ for all $i = 1, 2, \ldots, N$. Therefore, by applying a similar argument as above together with (14), it follows that $s_* \in S_i$ for every $i = 1, 2, \ldots, N$. Consequently, we conclude that $s_* \in S$.

Now, suppose that there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ that converges weakly to some point $s^\dagger \in S$. We aim to prove that $s^\dagger = s_*$. Following a similar line of reasoning as above, we conclude that $s^\dagger \in S$. Since both $s_*$ and $s^\dagger$ are elements of $S$, the limits $\lim_{n\to\infty} \|x_n - s_*\|^2$ and $\lim_{n\to\infty} \|x_n - s^\dagger\|^2$ exist and are finite. Therefore, the difference of these two limits also exists and is given by

$$\lim_{n\to\infty} \left(\|x_n - s_*\|^2 - \|x_n - s^\dagger\|^2\right) = 2 \lim_{n\to\infty} \langle x_n, s^\dagger - s_*\rangle + \|s_*\|^2 - \|s^\dagger\|^2.$$

This implies that the limit $\lim_{n\to\infty} \langle x_n, s^\dagger - s_*\rangle$ exists.

Moreover, since $x_{k_n} \rightharpoonup s_*$ and $x_{l_n} \rightharpoonup s^\dagger$, we obtain $\langle s_*, s^\dagger - s_*\rangle = \langle s^\dagger, s^\dagger - s_*\rangle$. Subtracting both sides, we deduce that $\|s_* - s^\dagger\|^2 = 0$, and hence $s_* = s^\dagger$. Therefore, the sequence $\{x_n\}$ converges weakly to $s_*$.

This completes the proof. □

**Remark 3.4.** To ensure that the conditions (C1) and (C2) in Theorem 3.3 are satisfied, one can choose the sequence $\{\alpha_{j,n}\}$ as following:

$$\alpha_{j,n} = \frac{1}{(n+1)^{1+\delta}\left(\sum_{j=1}^m \|x_{n+1-j} - x_{n-j}\|^2 + 1\right)}, \quad \text{for all } j = 1, 2, \ldots, m, \text{ where } \delta > 0.$$

**Remark 3.5.** In the case $m = 1$, our new algorithm reduces to Algorithm 3.1 in [10] of Reich et al.

**Remark 3.6.** Let $\mathcal{X}$ and $\mathcal{Y}_i$ $(i = 1, 2, \ldots, N)$ be Hilbert spaces.

(i) Assume that $T : \mathcal{X} \to \mathcal{X}$ and $T_i : \mathcal{Y}_i \to \mathcal{Y}_i$ are nonexpansive mappings, and let $h = 0$ and $h_i = 0$ for all $i = 1, 2, \ldots, N$. If in Algorithm 2, we choose $F = I - T$ and $F_i = I_i - T_i$, where $I$ and $I_i$ denote the identity operators on $\mathcal{X}$ and $\mathcal{Y}_i$, respectively, then we obtain algorithms for solving the *split common fixed point problem with multiple output sets*:

Find $p \in \mathcal{X}$ such that $T(p) = p$ and $T_i(T_i p) = T_i p$ for all $i = 1, 2, \ldots, N$.

In particular, if $T = P_C$ and $T_i = P_{Q_i}$, where $C \subset \mathcal{X}$ and $Q_i \subset \mathcal{Y}_i$ are nonempty closed convex sets, we obtain algorithms for solving the *split feasibility problem with multiple output sets*.

(ii) Let $B : \mathcal{X} \to 2^{\mathcal{X}}$ and $B_i : \mathcal{Y}_i \to 2^{\mathcal{Y}_i}$ be maximal monotone operators, and suppose that $h = 0$ and $h_i = 0$ for all $i = 1, 2, \ldots, N$. If in Algorithm 2, we set $F = I - J_r^B$ and $F_i = I_i - J_{r_i}^{B_i}$, where $J_r^B = (I + rB)^{-1}$ and $J_{r_i}^{B_i} = (I_i + r_i B_i)^{-1}$ are the resolvent operators of $B$ and $B_i$, respectively, then we obtain algorithms for solving the *split common null point problem with multiple output sets*:

Find $p \in \mathcal{X}$ such that $B(p) \ni 0$ and $B_i(T_i p) \ni 0$ for all $i = 1, 2, \ldots, N$.

**Example 3.7.** In this example, the algorithm is implemented in MATLAB R2014a running on the LAPTOP DELL-PRECISION 5560, 11th Gen Intel(R) Core(TM) i7-11850H @ 2.50GHz(16 CPUs) and 16GB RAM.

We consider the split minimum point problem with multiple output sets with the following data. Let $f$, $f_1$, $f_2$ and $f_3$ be convex functions on $\mathbb{R}^4$, $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively, which are defined as follows

$$f(x) = \frac{1}{2}(x_1 + x_2 + 3x_3 - 2x_4 - 1)^2 \text{ for all } x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^4,$$
$$f_1(y) = 2y_1^2 + \frac{y_2^2}{2} + 2y_1y_2 \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2,$$
$$f_2(z) = \frac{1}{2}(z_1 - 2z_2 + z_3 - 1)^2 \text{ for all } z = (z_1, z_2, z_3) \in \mathbb{R}^3,$$
$$f_3(t) = \frac{1}{2}(3t_1 + 2t_2 - t_3 + t_4 - 2)^2 \text{ for all } t = (t_1, t_2, t_3, t_4) \in \mathbb{R}^4.$$

The representing matrices of the transfer mappings $M_1 : \mathbb{R}^4 \to \mathbb{R}^2$, $M_2 : \mathbb{R}^4 \to \mathbb{R}^3$ and $M_3 : \mathbb{R}^4 \to \mathbb{R}^4$ are

$$M_1 = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & -2 & 6 \end{pmatrix}, M_2 = \begin{pmatrix} 3 & -4 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 6 & 4 & -4 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & 0 & 8 \\ -4 & 2 & 3 & -4 \end{pmatrix},$$

respectively.
The problem is to find an element $x^* \in \mathbb{R}^4$ such that

$$\begin{aligned} &x^* \in \arg\min_{x\in\mathbb{R}^4} f(x),\ M_1x^* \in \arg\min_{y\in\mathbb{R}^2} f_1(y), \\ &M_2x^* \in \arg\min_{z\in\mathbb{R}^3} f_2(z),\ M_3x^* \in \arg\min_{t\in\mathbb{R}^4} f_3(t), \end{aligned} \tag{16}$$

It is not difficult to check that $x^* \in \Omega$, where

$$\Omega = \{(1 - 5a + 6b, 2a - 4b, a, b) : a, b \in \mathbb{R}\},$$

and $f, f_1, f_2, f_3$ are Fréchet differentiable convex functions, $\nabla f, \nabla f_i$ are Lipschitz continuous, $M_j : \mathbb{R}^4 \to \mathbb{R}^j$, $j = 2, 3, 4$, be bounded linear operators. It follows from Baillon–Haddad theorem that $\nabla f$ and $\nabla f_i, i = 1, 2, 3, 4$, are inverse strongly monotones.

Hence, we can apply Algorithm 3.2 to $F = \nabla f, F_i = \nabla f_i, h = h_1 = h_2 = h_3 = 0$ to solve Problem (16). We now illustrate the convergence of Algorithm 3.2 in the case where the iterative formula involves two inertial terms, that is $m = 2$. Now, we test the convergence of the iterative method (2) with the initial points set as $x_{-2} = x_{-1} = x_0 = (1, 2, 3, 4)$ and different choices of inertial parameters. Based on Remark 4.1 in [10], we use the condition $\epsilon_n := \|x_n - y_{n-1}\| < \varepsilon$ to stop the iterative process. Let $\sigma_n = \max_i\{\|\nabla f(x_n)\|, \|\nabla f_i(T_ix_n)\|\},\ i = 1, 2, 3$, we obtain the numerical results as shown in Table 1.

**Table 1.** *Table of numerical results for Algorithm 3.2*

| $\varepsilon$ | $\alpha_{1,n} = 0.1, \alpha_{2,n} = -0.005$ | | | $\alpha_{1,n} = 0.1, \alpha_{2,n} = 0$ | | | $\alpha_{1,n} = \alpha_{2,n} = 0$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $n$ | Time (s) | $\sigma_n$ | $n$ | Time (s) | $\sigma_n$ | $n$ | Time (s) | $\sigma_n$ |
| $10^{-5}$ | 7 | 0.0005 | $5.9482 \times 10^{-7}$ | 7 | 0.0011 | $5.9482 \times 10^{-7}$ | 8 | 0.0007 | $1.4260 \times 10^{-6}$ |
| $10^{-6}$ | 8 | 0.0005 | $4.0319 \times 10^{-8}$ | 8 | 0.0005 | $4.8026 \times 10^{-7}$ | 9 | 0.0008 | $1.8556 \times 10^{-7}$ |
| $10^{-7}$ | 8 | 0.0006 | $4.0319 \times 10^{-8}$ | 9 | 0.0005 | $1.6419 \times 10^{-8}$ | 10 | 0.0010 | $2.4255 \times 10^{-8}$ |
| $10^{-8}$ | 9 | 0.0006 | $4.2228 \times 10^{-9}$ | 10 | 0.0006 | $3.5875 \times 10^{-9}$ | 11 | 0.0011 | $3.1777 \times 10^{-9}$ |

**Remark 3.8.** The results in Table 1 indicate that Algorithm 3.2 with two inertial terms ($\alpha_{1,n} = 0.1$,

$\alpha_{2,n} = -0.005$) converges more rapidly than both the single-inertial version ($\alpha_{1,n} = 0.1$, $\alpha_{2,n} = 0$), corresponding to Algorithm 3.1 in [10], and the classical proximal point algorithm without inertia ($\alpha_{1,n} = \alpha_{2,n} = 0$). This demonstrates the advantage of employing both inertial parameters to accelerate convergence.

## 4. Conclusion

This paper investigates the SCSP-MOS problem, a generalized split feasibility problem, and proposes an iterative method with weak convergence to a solution, as established in Theorem 3.3 under suitable conditions. The numerical example shows that the algorithm achieves better convergence results with more inertial terms. The algorithm extends existing methodologies and provides a versatile tool for addressing complex models in applied mathematics and optimization.

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