ON THE MEAN VALUE THEOREM AND ROLLE'S THEOREM FOR FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT

The classical forms of Rolle's Theorem and Lagrange's Mean Value Theorem for a differentiable, single-valued real function, are fundamental results in mathematical analysis, with many important applications. A natural and interesting question is to extend these theorems to the case of functions of several variables. In this paper, we present a version of the Mean Value Theorem for functions of several variables and provide an application of the classical Mean Value Theorem in functional equations. When extending the mean value theorem from the one-variable case to several variables, the function on an interval is replaced by a function defined on the closure of a domain, and the endpoint values are replaced by the values on the boundary of that domain. To prove our mean value theorem for differentiable functions of several real variables, we make use of a version of Rolle's Theorem due to Alberto Fiorenza and Renato Fiorenza (2024).

VỀ ĐỊNH LÝ GIÁ TRỊ TRUNG BÌNH VÀ ĐỊNH LÝ ROLLE CHO HÀM NHIỀU BIẾN

Lâm Trần Phương Thủy
Trường Đại học Điện lực

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TỪ KHÓA

Định lí Rolle
Định lí Lagrange
Đạo hàm cổ điển
Hàm tuyến tính
Giải tích thực

TÓM TẮT

Định lý Rolle và Định lý giá trị trung bình của Lagrance đối với hàm khả vi thực một biến là những kết quả nền tảng trong giải tích và có nhiều ứng dụng quan trọng. Một câu hỏi tự nhiên và thú vị là mở rộng các kết quả này cho trường hợp hàm nhiều biến. Trong bài báo này, chúng tôi đưa ra một sự mở rộng của Định lý giá trị trung bình cho các hàm nhiều biến và một ứng dụng của Định lí này đối với phương trình hàm. Khi chuyển từ trường hợp một biến sang trường hợp nhiều biến, hàm số trên đoạn thẳng được thay thế bởi hàm số xác định trên tập đóng của một miền, và các giá trị tại hai đầu mút được thay bởi các giá trị trên biên của miền đó. Để chứng minh Định lý giá trị trung bình cho các hàm khả vi nhiều biến thực, chúng tôi sử dụng một phiên bản của Định lý Rolle do Alberto Fiorenza và Renato Fiorenza thiết lập (2024).

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1. Introduction

Rolle's theorem captures a fundamental geometric intuition: if a smooth curve begins and ends at the same height over a closed interval, then somewhere in between, it must "flatten out"—its instantaneous rate of change vanishes. Formally, this means that a function continuous on a closed interval and differentiable inside must attain a zero derivative at some interior point when its values at the endpoints coincide.

An essential consequence of Rolle's theorem is the Mean Value Theorem, formulated by Lagrange. It states that if a function is continuous on a closed interval [a,b] and differentiable on the open interval (a,b), then there exists a point c in (a,b) such that the derivative of the function at c equals the average rate of change of the function over the interval, i.e.,

$$\frac{g(a)-g(b)}{a-b}=g'(c).$$

Alberto et al. [1] extended the classical Rolle's theorem to a broader setting, encompassing real-valued functions defined on domains within locally convex, separated topological vector spaces, including those of infinite dimension.

To present their result, we first recall the classical definition of the Gâteaux derivative, which serves as a generalization of the directional derivative in the context of differential calculus. Let X be a topological separated vector space, $D\subset X$ an open subset, and g a real-valued continuous function defined on D. For each point $a\in D$, the function g is said to be Gâteaux differentiable at a if, for every direction $h\in X$, the following limit exists and is finite:

$$\lim_{\lambda\to 0}\frac{g(a+\lambda h)-g(a)}{\lambda}.$$

This defines the Gâteaux derivative of the function g at a in the direction h.

In [1], the authors obtained the following version of Rolle's theorem.

Theorem A. Let X be a vector space endowed with a locally convex and separated topology. Suppose $D\subset X$ is a proper, closed subset whose interior is nonempty. Consider a real-valued function g that is continuous on D and possesses a Gateaux derivative at every point within the interior of D. Assume further that g takes a constant value along the boundary of D, and that both the minimum and maximum of g exist on D (as is guaranteed, for example, when D is compact). Then there must exist at least one interior point of D at which the Gateaux derivative of g vanishes.

In this paper, using the results documented by [1] as mentioned above, we present a version of the Mean Value Theorem for several variables. An application of the Mean Value Theorem in differential equations is also provided. For additional results concerning Rolle's Theorem and the Mean Value Theorem, we refer readers to [1] – [7].

2. Methods

Techniques from real analysis–particularly the result by Alberto Fiorenza and Renato Fiorenza ([1], Theorem 1), together with the classical mean value theorem for single-variable functions and fundamental properties of differentiable and continuous functions are employed in this study.

3. Results

The following is a version of the Mean Value Theorem for functions of several variables.

Theorem 3.1. *Let* Ω *be a bounded domain in* $R^n \subset R^{n+1}$ *and let* f *be a real function defined in* $\overline{\Omega}$ *, continuous in* $\overline{\Omega}$ *and differentiable in* Ω. *Let* $\Pr_{n+1}: R^{n+1} \to R^n$ *be the projection defined by* $\Pr_{n+1}(x_1,...,x_n,x_{n+1}) = (x_1,...,x_n)$. Denote by Γ_f the graph of f in R^{n+1}

$$\Gamma_f = \{(x_1,...,x_n, f(x_1,...x_n)):\ x=(x_1,...,x_n)\in\overline{\Omega}\}.$$

Assume that $\Pr_{n+1}^{-1}(\partial\Omega)\cap\Gamma_f$ *is a subset of a hyperplane* α *defined by* $x_{n+1} = a_1x_1 + \cdots + a_nx_n + a$. *Then there exists* $c=(c_1,...,c_n)\in\Omega$ *such that*

$$\left(\frac{\partial f}{\partial x_1}(c),...,\frac{\partial f}{\partial x_n}(c)\right) = (a_1,...,a_n).$$

Proof.

Denote by h the function in $\overline{\Omega}$ defined by

$$h(x_1,...,x_n) = a_1x_1 + \cdots + a_nx_n - f(x_1,...,x_n) + a.$$

Then h is *continuous in* Ω *and differentiable in* Ω. *On the other hand* $h=0$ *on* $\partial\Omega$. Therefore, by Theorem A, there exists $(c_1,...c_n)\in\Omega$ such that

$$\left(\frac{\partial h}{\partial x_1}(c),...,\frac{\partial h}{\partial x_n}(c)\right) = (0,...,0) \text{ and hence, } \left(\frac{\partial f}{\partial x_1}(c),...,\frac{\partial f}{\partial x_n}(c)\right) = (a_1,...,a_n).$$

An application of the classical Mean Value Theorem to functional equations involving functions of several variables is presented as follows.

Theorem 3.2. Let $f: R^n \to R$ be a differentiable function and let a real number $\alpha\in(0;1)$. Assume that for each $i\in\{1,...,n\}$, $f(x_1,...,x_n) = f(x_1,...,\alpha x_i,...,x_n) + (1-\alpha)x_i\dfrac{\partial f}{\partial x_i}(x)$

for all $x=(x_1,...x_n)\in R^n$. Then the function f is linear, and can thus be represented as $f(x) = a_1x_1 + \cdots + a_nx_n + a$, where the coefficients $a_1,...a_n, a$ are real constants.

Proof. First, we prove that for each $i\in\{1,...,n\}$, $\dfrac{\partial f}{\partial x_i}(x)$ is continuous in the variable x_i.

It finds

$$\lim_{t\to 0}\frac{\partial f}{\partial x_i}(x_1,...,x_{i-1},t,x_{i+1},...,x_n) = \lim_{x\to 0}\frac{f(x_1,...,x_{i-1},t,x_{i+1},...,x_n) - f(x_1,...x_{i-1},\alpha t,x_{i+1},...,x_n)}{(1-\alpha)t}$$

$$= \frac{\partial f}{\partial x_i}(x_1,...x_{i-1},0,x_{i+1},...,x_n).$$

Therefore, $\dfrac{\partial f}{\partial x_i}(x)$ is continuous in the variable x_i at 0. It is obviously continuous in the variable x_i wherever $x_i\neq 0$, since it is the ratio of two continuous functions,

$$\frac{\partial f}{\partial x_i}(x_1,...x_{i-1},x_i,x_{i+1},...,x_n) = \frac{f(x_1,...x_{i-1},x_i,x_{i+1},...,x_n) - f(x_1,...x_{i-1},\alpha x_i,x_{i+1},...,x_n)}{(1-\alpha)x_i}.$$

Consider a fixed a point $c=(c_1,...,c_n)\in R^n$.

Case 1. $c_i \geq 0.$

Let M denote the set of all nonnegative constants t satisfying the following condition:

$$\frac{\partial f}{\partial x_i}(c_1,...,t,...c_n) = \frac{\partial f}{\partial x_i}(c_1,...,c_i,...c_n).$$

Clearly, M is non-empty and has a lower bound. Hence, the infimum t_i of M exists.

Since $\frac{\partial f}{\partial x_i}(x)$ is continuous in the variable x_i, M is closed, and hence $t_0 \in M.$

Assume that $t_i > 0.$ By the assumption, we have

$$\frac{\partial f}{\partial x_i}(c_1,...,t_i,...c_n) = \frac{f(c_1,...,t_i,...c_n) - f(c_1,...,\alpha t_i,...c_n)}{(1-\alpha)t_i}. \quad (1)$$

By applying the Mean Value Theorem to the function $f(c_1,...,t,...c_n)$ with the variable t over the interval $[\alpha t_i;t_i]$, we can guarantee the existence of a value $\beta \in (\alpha t_i;t_i)$ such that

$$\frac{f(c_1,...,t_i,...c_n) - f(c_1,...,\alpha t_i,...c_n)}{(1-\alpha)t_i} = \frac{\partial f}{\partial x_i}(c_1,...,\beta,...c_n). \quad (2)$$

From (1) and (2), we get that $\beta \in M$, and hence, $\beta < t_i = \min\{t : t \in M\};$ this is impossible. Hence, $t_i = 0.$ This implies that

$$\frac{\partial f}{\partial x_i}(c_1,...,c_i,...c_n) = \frac{\partial f}{\partial x_i}(c_1,...,0,...c_n).$$

Hence, $\frac{\partial f}{\partial x_i}(x)$ is constant in the variable, for all $i \in \{1,...,n\}.$ This implies that f is a linear function.

Case 2. $c_i \leq 0.$

Let M' denote the set of all constants $t \leq 0$ satisfying the following condition:

$$\frac{\partial f}{\partial x_i}(c_1,...,t,...c_n) = \frac{\partial f}{\partial x_i}(c_1,...,c_i,...c_n).$$

Denote by t_i^* be the supremum of $M'.$

Analogous to Case 1, it can be deduced that $t_i^* \in M'$, and by applying the Mean Value Theorem on the interval $[t_i^*;\alpha t_i^*]$, $t_i^* = 0.$ Therefore,

$$\frac{\partial f}{\partial x_i}(c_1,...,c_i,...c_n) = \frac{\partial f}{\partial x_i}(c_1,...,0,...c_n).$$

From both Case 1 and Case 2, it can be concluded that, $\frac{\partial f}{\partial x_i}(c_1,...,t,...c_n)$ is constant in the variable t, for all $i \in \{1,...,n\}$ and constants $c_1,...,c_{i-1},c_{i+1},...,c_n.$ Therefore,

$$\frac{\partial f}{\partial x_i}(c_1,...,c_n) = \frac{\partial f}{\partial x_i}(0,...,0)$$

for all $c = (c_1, ..., c_n) \in R^n$.

This implies that f is a linear function.

4. Conclusion

This paper establishes a generalized version of the mean value theorem for functions of several variables and presents an application of the classical mean value theorem to functional equations involving multivariable functions. While the results contribute to the theoretical understanding of mean value-type results in higher dimensions, the current work is limited to functions defined on bounded subsets of R^n and taking values in R, under relatively strong smoothness assumptions.

A promising direction for future research is to develop a mean value theorem for differentiable mappings from bounded subsets of R^n into R^m, possibly under weaker regularity conditions or more general geometric settings.

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