

THE EXISTENCE OF WEAK SOLUTION TO FRACTIONAL ANISOTROPIC EQUATIONS IN \mathbb{R}^N WITH EXPONENTIAL GROWTH

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ARTICLE INFO	ABSTRACT
Received: 07/5/2025	In this paper, we investigate the existence of a weak solution to a fractional anisotropic equation with a Choquard reaction and exponential growth. The variational methods are used to study above problem. Namely, Lions's result and Mountain Pass Theorem are combined to get the existence of a weak solution to our equation. Under some suitable conditions on nonlinear reaction, studies show that the energy function to our problem satisfies the geometric condition of the Mountain Pass Theorem. Then there exists the Palais-Smale sequence with a positive level. Next, for that given condition on the nonlinear function, the mountain pass level is proved small enough, as a result, the fractional Trudinger-Moser inequality can be applied to get the compact of that Palais-Smale sequence. The weak limit of the Palais-Smale sequence is the weak solution of our problem. Some difficulties needed to be solved of estimating the mountain pass level, as well as the compact of the Palais-Smale sequence due to the Choquard reaction and exponential nonlinearity. In the future work, our results can be applied to study the multiplicity of weak solutions for the autonomous problem associated with our problem.
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SỰ TỒN TẠI NGHIỆM YẾU CỦA PHƯƠNG TRÌNH PHÂN THỨ KHÔNG ĐẲNG HƯỚNG TRONG \mathbb{R}^N VỚI ĐỘ TĂNG MŨ

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THÔNG TIN BÀI BÁO	TÓM TẮT
Ngày nhận bài: 07/5/2025	Bài báo này nghiên cứu sự tồn tại của nghiệm yếu cho một phương trình phân thứ bất đẳng hướng với hàm phi tuyến Choquard có độ tăng mũ. Phương pháp biến phân đã được sử dụng trong nghiên cứu này. Cụ thể, kết quả kiểu Lions và Định lý Vượt núi được kết hợp để chứng minh sự tồn tại của nghiệm yếu cho phương trình đã cho. Dưới một số điều kiện phù hợp đối với hàm phi tuyến, nghiên cứu chỉ ra rằng hàm năng lượng của bài toán thỏa mãn điều kiện hình học của Định lý Vượt núi. Từ đó tồn tại một dãy Palais-Smale với mức năng lượng dương. Tiếp theo, với điều kiện đã cho đối với hàm phi tuyến, mức vượt núi được chứng minh là đủ nhỏ, và do đó có thể áp dụng bất đẳng thức Trudinger-Moser phân thứ để thu được tính compact của dãy Palais-Smale. Giới hạn yếu của dãy Palais-Smale sẽ là nghiệm yếu của bài toán đặt ra. Một vài khó khăn cần được giải quyết của việc ước lượng mức vượt núi cũng như tính compact của dãy Palais-Smale do phương trình chứa số hạng phi tuyến Choquard và hàm phi tuyến có độ tăng mũ. Trong các công trình tương lai, kết quả của nghiên cứu này có thể được áp dụng để nghiên cứu tính đa nghiệm yếu cho bài toán không ô-tô-nôm liên kết với bài toán chúng tôi đang xét.
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1. Introduction

In this paper, the existence of weak solution to equation in \mathbb{R}^N is investigated

$$\sum_{i=1}^m \left((-\Delta)_{p_i}^s u + \eta_1 |u|^{p_i-2} u \right) = \eta_2^2 \left[\frac{1}{|x|^\mu} * H_1(u) \right] h_1(u) + \eta_3 h_2(u), \tag{1}$$

where $H_j(t) = \int_0^t h_j(t) dt$, $j=1,2$, $2 \leq p_1 = p < \dots < p_m < +\infty$, $0 < s < 1$, $N = ps$, $0 < \mu < N$,

$(-\Delta)_{p_i}^s$ ($i=1, \dots, m$) is a fractional p_i -Laplace operator which is defined along enough smooth function (up to a normalization constant) by

$$(-\Delta)_{p_i}^s \varphi(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p_i-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+p_i s}} dy$$

for $x \in \mathbb{R}^N$, where $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $B_\varepsilon(x)$ is a ball with center x and radius ε ; $\eta_1, \eta_2 > 0$ and $\eta_3 > 0$ are constants. We denote $\mathcal{D} := \{h : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is continuous function}\}$ and verifies the conditions as follows:

(h₁) The nonlinear reaction h is continuous and $h(t) = 0$ on $(-\infty, 0]$. For each $q_1 \geq p_m$, $q_2 \geq 1$ and there are positive real numbers $a_1 > 0$, $a_2 > 0$ so that

$$h(t) \leq a_1 |t|^{q_1-1} + a_2 \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) |t|^{q_2-1},$$

where $\Phi_{N,s_1}(y) = e^y - \sum_{j=0}^{j_p-2} \frac{y^j}{j!}$, $j_p = \min\{j \in \mathbb{N} : j \geq p\}$ and $\alpha_* < \alpha_{s,N}^*$, where

$$\alpha_{s,N}^* = N \left(\frac{2(N\omega_N)^2 \Gamma(p+1)}{N!} \sum_{k=0}^{\infty} \frac{(N+k-1)!}{k!} \frac{1}{(N+2k)^p} \right)^{\frac{s}{N-s}}$$

(h₂) $\lim_{t \rightarrow 0^+} \frac{h(t)}{t^{p_m-1}} = 0$.

(h₃) There is $\theta > p_m$ so that $2h(t) \geq h(t) \geq \theta H(t) > 0$ for all $t > 0$, where $H(t) = \int_0^t h(\tau) d\tau$.

(h₄) There exists $\gamma_1 > 0$ sufficiently large so that $H(t) \geq \gamma_1 |t|^\theta$ on $(0, \infty)$, where $\theta > p_m$ is a constant given in the condition (h₃).

We assume that two nonlinear reactions h_1 and h_2 belong to \mathcal{D} . Our work is motivated by some works in recently. In 2024, Ambrosio [1] and Zhang et al. [2] studied the multiple concentration solutions to problem

$$-\Delta_p^s u + (-\Delta)_q^s u + V(x) |u|^{p-2} u + |u|^{q-2} u = \left(\frac{1}{|x|^\mu} * F(u) \right) f(u) \tag{2}$$

where f has subcritical growth. Using Variational method, Nehari manifold and Ljusternik-Schnirelmann theory, they established the multiplicity and concentration of solution to (2). In 2024, Liu and Perera [3] studied the existence of weak solution to double phases equation in \mathbb{R}^N as

follows $-\Delta_p u - \Delta_q u = \lambda h(x) |u|^{r-2} u + g(x) |u|^{p^*-2} u$, via concentration compactness principle of Lions, where $1 < q \leq p < r < p^*$. Some more results on double phases equation with Berestycki-Lions type nonlinearity, and related equation we refer the readers to [4] - [9]. In addition, the authors in [10] studied the existence of weak solution to fractional anisotropic equation in \mathbb{R}^N .

For each $\eta > 0$, we use another norm on $W^{s,p_i}(\mathbb{R}^N)$ by $\|u\|_{p_i,\eta} = \eta \|u\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + u^{p_i}_{s,p_i}$. Here, for any $r > 1$, $L^r(\mathbb{R}^N)$ is Lebesgue Space of measurable functions in \mathbb{R}^N and $\|u\|_{L^r(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^r dx\right)^{\frac{1}{r}}$.

Denote $\mathfrak{M} = \bigcap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$, then \mathfrak{M} is a Banach space with norms equivalent $\|u\|_{\mathfrak{M}} = \sum_{i=1}^m \|u\|_{p_i}$ and $\|u\|_{\eta,\mathfrak{M}} = \sum_{i=1}^m \|u\|_{p_i,\eta}$, where $\|u\|_{p_i} = \|u\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + u^{p_i}_{s,p_i}$. Therefore, the best constant $\mathcal{A}_{\nu,\eta} > 0$ is well defined:

$$\mathcal{A}_{\nu,\eta} = \inf_{u \neq 0, u \in W^{s,N/s}(\mathbb{R}^N)} \frac{\|u\|_{\eta,\mathfrak{M}}}{\|u\|_{L^\nu(\mathbb{R}^N)}}, \text{ for all } \nu \in \left[\frac{N}{s}, +\infty\right).$$

Furthermore, it holds $\|u\|_{L^\nu(\mathbb{R}^N)} \leq \mathcal{A}_{\nu,\eta}^{-1} \|u\|_{\eta,\mathfrak{M}}$ for all $u \in \mathfrak{M}$. (3)

The energy functional $\mathcal{J}_{\eta_1,\eta_2,\eta_3} : \mathfrak{M} \rightarrow \mathbb{R}$ of problem (1) which has the form

$$\begin{aligned} \mathcal{J}_{\eta_1,\eta_2,\eta_3}(u) &= \sum_{i=1}^m \frac{1}{p_i} \|u\|_{p_i,\eta_1}^{p_i} - \frac{\eta_2^2}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{H_1(u(y))H_1(u(x))}{|x-y|^\mu} dx dy - \eta_3 \int_{\mathbb{R}^N} H_2(u) dx \\ &= \sum_{i=1}^m \frac{1}{p_i} \|u\|_{p_i,\eta_1}^{p_i} - \frac{\eta_2^2}{2} \int_{\mathbb{R}^N} \mathcal{G} u(x) H_1 u(x) dx - \eta_3 \int_{\mathbb{R}^N} H_2 u dx \end{aligned} \quad (4)$$

where $\mathcal{G} u(x) = \int_{\mathbb{R}^N} \frac{H_1 u(y)}{|x-y|^\mu} dy$. Then, we find that the weak solution of (1) is a critical point

of energy function in \mathfrak{M} . Our main result is stated as follows:

Theorem 1. Assume that (h₁) - (h₄) hold, then problem (1) attains a nontrivial nonnegative weak solution in \mathfrak{M} .

2. Some preliminary results

To prove main result, we need the following lemma:

Lemma 1 [6]. Given $r, t > 1$ and $0 < \mu < N$ so that $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{t} = 2$, $f \in L^r(\mathbb{R}^N)$ and $g \in L^t(\mathbb{R}^N)$. Then there is a sharp constant $C(r, N, \mu, t) > 0$ independent of f and g so that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\mu} dx dy \leq C(r, N, \mu, t) \|f\|_{L^r(\mathbb{R}^N)} \|g\|_{L^t(\mathbb{R}^N)}.$$

In the application, we use $r = t$ in Lemma 1, it means that $\frac{2}{t} + \frac{\mu}{N} = 2$. Then if we choose

$H u = |u|^q$, it holds that $\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * H u \right] H u dx$ is well-defined on $L^t(\mathbb{R}^N)$ with $t = \frac{2N}{2N - \mu}$.

Due to the continuously embedded $W^{s,N/s} \mathbb{R}^N$ into $L^r \mathbb{R}^N$, for all $r \in \left[\frac{N}{s}, +\infty \right)$, we need

$qt \geq \frac{N}{s}$, then $q \geq \frac{N}{st} = \frac{2N-\mu}{2s}$. Using fractional Trudinger Moser inequality [7] and $C_0^\infty \mathbb{R}^N$ is a density subspace of $W^{s,p} \mathbb{R}^N$, we know that $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$ is well defined on $\mathfrak{M} \subset W^{s,N/s} \mathbb{R}^N$.

Lemma 2. *The functional $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$ satisfies the following results:*

(a) *There are $\zeta_*, \rho_0 > 0$ satisfying $\mathcal{J}_{\eta_1, \eta_2, \eta_3} u \geq \rho_0$ with $\|u\|_{\eta_1, \mathfrak{M}} \mathbb{R}^N = \zeta_*$.*

(b) *There is an element $v \in C_0^\infty \mathbb{R}^N$ with $\|v\|_{\eta_1, \mathfrak{M}} > \zeta_*$, so that $\mathcal{J}_{\eta_1, \eta_2, \eta_3} v < 0$.*

Proof. (a) By (h₁), there are $b_1 > 0, b_2 > 0$ so that

$$h_j t \leq b_1 |t|^{q_1-1} + b_2 \Phi_{N,s} \alpha_0 |t|^{N/N-s} |t|^{q_2-1}, \text{ for each } q_1 \geq p_m > \frac{N}{s} > \frac{2N-\mu}{s}, q_2 > p_m.$$

Hence, we deduce $|H_j t| \leq b_1 |t|^{q_1} + b_2 |t|^{q_2} \Phi_{N,s} \alpha_0 |t|^{N/N-s}$ for all $t \in \mathbb{R}, j=1,2$.

Apply Lemma 1 and by arguments [8, Lemma 3], for some $t > 1$, there are two positive constants \mathcal{D} and D_1 so that

$$\int_{\mathbb{R}^N} \mathcal{G} u x H_1 u x dx \leq D_1 \|u\|_{L^{\frac{2Nq_1}{2N-\mu}} \mathbb{R}^N}^{2q_1} + \mathcal{D} \|u\|_{L^{\frac{2Nq_2}{2N-\mu}} \mathbb{R}^N}^{2q_2} \text{ for } \|u\|_{\eta_1, \mathfrak{M}} \text{ sufficiently small.}$$

Similar, for any $\zeta > 0$, there exists $C \zeta > 0$ so that

$$h_2 t \leq \zeta |t|^{p_m-1} + C \zeta \Phi_{N,s} \alpha_0 |t|^{N/N-s} |t|^{q_2-1} \tag{5}$$

for some $q_2 > p_m$. Hence, using Trudinger Moser inequality for $\|u\|_{\eta_1, \mathfrak{M}}$ small enough, there exists $D(\zeta) > 0$ such that we also have

$$\int_{\mathbb{R}^N} H_2 u \leq \zeta \|u\|_{L^{p_m} \mathbb{R}^N}^{p_m} + D \zeta \|u\|_{L^{q_2} \mathbb{R}^N}^{q_2}. \tag{6}$$

Combine (5) and (6), for $\|u\|_{\eta_1, \mathfrak{M}}$ small enough, we get

$$\begin{aligned} \mathcal{J}_{\eta_1, \eta_2, \eta_3} u \geq & \frac{m^{1-p_m}}{p_m} \|u\|_{\eta_1, \mathfrak{M}}^{p_m} - \frac{\eta_2^2}{2} \left(D_1 \mathcal{A}_{\frac{2Nq_1}{2N-\mu}, \eta_1}^{-2q_1} \|u\|_{\eta_1, \mathfrak{M}}^{2q_1} - \mathcal{D} \mathcal{A}_{\frac{2Nq_2}{2N-\mu}, \eta_1}^{-2q_2} \|u\|_{\eta_1, \mathfrak{M}}^{2q_2} \right) \\ & - \eta_3 \zeta \mathcal{A}_{p_m, \eta_1}^{-p_m} \|u\|_{\eta_1, \mathfrak{M}}^{p_m} - D \zeta \mathcal{A}_{q_2, \eta_1}^{-q_2} \|u\|_{\eta_1, \mathfrak{M}}^{q_2}. \end{aligned} \tag{7}$$

Then if we choose $\zeta > 0$ small enough so that $\frac{m^{1-p_m}}{p_m} - \eta_3 \zeta \mathcal{A}_{p_m, \eta_1}^{-p_m} > 0$, then with $t \geq 0$ by considering the function

$$\mathcal{H} t = \left(\frac{m^{1-p_m}}{p_m} - \eta_3 \zeta \mathcal{A}_{p_m, \eta_1}^{-p_m} \right) t^{p_m} - \frac{\eta_2^2}{2} \left(D_1 \mathcal{A}_{\frac{2Nq_1}{2N-\mu}, \eta_1}^{-2q_1} t^{2q_1} - \mathcal{D} \mathcal{A}_{\frac{2Nq_2}{2N-\mu}, \eta_1}^{-2q_2} t^{2q_2} \right) - \eta_3 - D \zeta \mathcal{A}_{p_2, \eta_1}^{-p_2} t^{q_2},$$

and computing as in [8, Lemma 3], there are $\zeta_*, \rho_0 > 0$ so that $\mathcal{J}_{\eta_1, \eta_2, \eta_3} u \geq \rho_0$ with $\|u\|_{\eta_1, \mathfrak{M}} \mathbb{R}^N = \zeta_*$.

(b) Choose $u_0 \in \mathfrak{M}$ so that $u_0 \geq 0$ in \mathbb{R}^N . By the condition (h₄), we get

$$\mathcal{J}_{\eta_1, \eta_2, \eta_3} t u_0 \leq \sum_{i=1}^m \frac{t^{p_i}}{p_i} \|u_0\|_{p_i, \eta_i}^{p_i} - \eta_3 t^\theta \int_{\mathbb{R}^N} |u_0|^\theta dx \rightarrow -\infty$$

as $t \rightarrow \infty$. Set $e = t u_0$ and t sufficiently large, we get $\mathcal{J}_{\eta_1, \eta_2, \eta_3} e < 0$.

Let $c \in \mathbb{R}$, and $u_n \subset \mathfrak{M}$ so that $\mathcal{J}_{\eta_1, \eta_2, \eta_3} u_n \rightarrow c$ and $\mathcal{J}'_{\eta_1, \eta_2, \eta_3} u_n \rightarrow 0$ in \mathfrak{M}^* as $n \rightarrow \infty$, then u_n is called *PS* sequence of $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$ with level c and it is denoted by PS_c , where \mathfrak{M}^* is dual space of \mathfrak{M} . From Lemma 2 and a version of Mountain Pass Theorem without the Palais-Smale condition, we obtain the $PS_{c_{\eta_1, \eta_2, \eta_3}}$ sequence $u_n \subset \mathfrak{M}$ with level $c_{\eta_1, \eta_2, \eta_3}$, where

$$c_{\eta_1, \eta_2, \eta_3} = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{J}_{\eta_1, \eta_2, \eta_3} \gamma t \quad \text{and} \quad \Gamma = \{ \gamma \in C[0, 1], \mathfrak{M} : \gamma(0) = 0, \mathcal{J}_{\eta_1, \eta_2, \eta_3} \gamma(1) < 0 \}.$$

Lemma 3. (i) Suppose that u_n is a $PS_{c_{\eta_1, \eta_2, \eta_3}}$ sequence for $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$. Then, there exists a constant C_{γ_1} satisfying that $\zeta_* \leq c_\eta \leq C_{\gamma_1}$, where

$$C_{\gamma_1} = \left(\sum_{j=1}^m a_j \right) \left(1 - \frac{N}{\theta s} \right) \left(\frac{N \sum_{j=1}^m a_j}{\theta s b} \right)^{N / \theta s - N} \tag{8}$$

with $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $a, b > 0$.

(ii) Suppose that u_n is a PS_{c_η} sequence for $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$ satisfying $\limsup_{n \rightarrow \infty} \|u_n\|_{\eta_1, \mathfrak{M}}^{N / N - s} < \frac{\alpha_*}{\alpha_0} \sigma^{s / N - s}, l > 1$, then $u_n \rightarrow u$ in \mathfrak{M} and $\mathcal{J}'_{\eta_1, \eta_2, \eta_3} u = 0$.

Proof. Choose $\varphi \in \mathfrak{M}$ such that $\varphi \geq 0$ in \mathbb{R}^N . Then we have

$$\mathcal{J}_{\eta_1, \eta_2, \eta_3} t \varphi \leq \sum_{i=1}^m \frac{t^{p_i}}{p_i} \|\varphi\|_{p_i, \eta_i}^{p_i} - \gamma_1 \eta_3 t^\theta \|\varphi\|_{L^\theta(\mathbb{R}^N)}^\theta. \tag{9}$$

We denote $a_j = \frac{\|\varphi\|_{p_j, \eta_j}^{p_j}}{p_j}, j = 1, \dots, m, b = \gamma_1 \eta_3 \|\varphi\|_{L^\theta(\mathbb{R}^N)}^\theta$ and $g(t) = \sum_{j=1}^m a_j t^{p_j} - b t^\theta$ on $0, +\infty$. It holds

$$c_{\eta_1, \eta_2, \eta_3} \leq \max_{t \in [0, 1]} g(t) + \max_{t \geq 1} g(t). \tag{10}$$

We first consider that $t \in [0, 1]$, then we get $g(t) \leq h(t) = \sum_{j=1}^m a_j t^{N/s} - b t^\theta$. By simple computing, we get

$$\max_{t \in [0, 1]} g(t) \leq h(\theta_{\gamma_1}) = C_{\gamma_1}, \tag{11}$$

where $\theta_{\gamma_1} = \left(\frac{N \sum_{j=1}^m a_j}{\theta s b} \right)^{s / \theta s - N} \leq 1$, if we choose $\gamma_1 \geq \frac{N \sum_{j=1}^m a_j}{\theta s \eta_3 \|\varphi\|_{L^\theta(\mathbb{R}^N)}^\theta} := \gamma_*$.

Compute directly, we get

$$C_{\gamma_1} = h(\theta_{\gamma_1}) = \left(\sum_{j=1}^m a_j \right) \left(1 - \frac{N}{\theta s} \right) \left(\frac{N \sum_{j=1}^m a_j}{\theta s b} \right)^{N / \theta s - N}. \tag{12}$$

We see that $\lim_{\gamma_1 \rightarrow +\infty} \theta_{\gamma_1} = 0$, then $\lim_{\gamma_1 \rightarrow +\infty} h_{\theta_{\gamma_1}} = 0$. Compute similarly as before, we deduce that for all $t \geq 1$, $g_t \leq h_* t = \sum_{j=1}^m a_j t^{p_j} - bt^\theta$ and h_* attains uniqueness local

maximum point at $\beta_{\gamma_1} = \left(\frac{p_m \sum_{j=1}^m a_j}{\theta b} \right)^{1/\theta - p_m}$ on $0, +\infty$. Now, if we take $\gamma_1 \geq \gamma_{**}$, where

$$\gamma_{**} = \frac{p_m \sum_{j=1}^m a_j}{\theta \eta_3 \|\varphi\|_{L^\theta \mathbb{R}^N}^\theta},$$

then it holds $\beta_{\gamma_1} \leq 1$. Hence, we obtain $\max_{t \geq 1} g_t \leq h_* 1 = \sum_{j=1}^m a_j - b$.

Set $\gamma_{***} = \frac{\sum_{j=1}^m a_j}{\eta_3 \|\varphi\|_{L^\theta \mathbb{R}^N}^\theta} > \gamma_{**} > \gamma_*$. Then we get

$$\max_{t \geq 1} g_t \leq 0 \text{ for all } \gamma_1 \geq \gamma_{***}. \tag{13}$$

Combine Lemma 2, (10), (11), (12) and (13), we have

$$\zeta_* \leq c_{\eta_1, \eta_2, \eta_3} \leq C_{\gamma_1} = \left(\sum_{j=1}^m a_j \right) \left(1 - \frac{N}{\theta s} \right) \left(\frac{N \sum_{j=1}^m a_j}{\theta s b} \right)^{s/\theta s - N} \tag{14}$$

for $\gamma_1 \geq \gamma_{***}$. Clearly, $c_{\eta_1, \eta_2, \eta_3}$ is small enough as γ_1 is large enough.

The following result is a version of Lions's result:

Lemma 4 [9]. *If u_n is a bounded sequence in $W^{s, N/s} \mathbb{R}^N$ and*

$$\limsup_{n \rightarrow \infty} \int_{B_R^N} |u_n(x)|^{N/s} dx = 0$$

for some $R > 0$, then $u_n \rightarrow 0$ in $L^q \mathbb{R}^N$ for all $q \in \left(\frac{N}{s}, +\infty \right)$.

3. Proof of Theorem 1

Proof. By Lemma 2, there is a $PS_{c_{\eta_1, \eta_2, \eta_3}}$ sequence u_n of the functional $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$ in \mathfrak{M} , it means that $\mathcal{J}_{\eta_1, \eta_2, \eta_3} u_n \rightarrow c_{\eta_1, \eta_2, \eta_3}$ and $\mathcal{J}'_{\eta_1, \eta_2, \eta_3} u_n \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, we have

$$c_{\eta_1, \eta_2, \eta_3} = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{J}_{\eta_1, \eta_2, \eta_3} \gamma t \quad \text{and}$$

$$\Gamma = \{ \gamma \in C[0, 1], \mathfrak{M} : \gamma(0) = 0, \mathcal{J}_{\eta_1, \eta_2, \eta_3} \gamma(1) < 0 \}.$$

By arguments as Lemma 3, we have

$$\begin{aligned} c_{\eta_1, \eta_2, \eta_3} + o_n(1) &= \mathcal{J}_{\eta_1, \eta_2, \eta_3} u_n = \mathcal{J}_{\eta_1, \eta_2, \eta_3} u_n - \frac{1}{\theta} \langle \mathcal{J}'_{\eta_1, \eta_2, \eta_3} u_n, u_n \rangle \\ &= \sum_{i=1}^m \left(\frac{1}{p_j} - \frac{1}{\theta} \right) \|u_n\|_{p_j, \eta_i}^{p_j} - \frac{\eta_2^2}{2\theta} \int_{\mathbb{R}^N} \frac{1}{|x|^\mu} * H_1 u \theta H_1 u x - 2h_1 u u dx \\ &\quad - \frac{\eta_3}{\theta} \int_{\mathbb{R}^N} \theta H_2 u - h_2 u u dx \geq \left(\frac{1}{p_1} - \frac{1}{\theta} \right) \|u_n\|_{p_1, \eta_1}^{p_1}. \end{aligned} \tag{15}$$

It implies that there exists n_0 so that

$$\left(\frac{1}{p_1} - \frac{1}{\theta}\right) \|u_n\|_{p, \eta_1}^p \leq c_{\eta_1, \eta_2, \eta_3} + \xi, p_1 = p \tag{16}$$

for all $n \geq n_0$.

$$\|u_n\|_{p, \eta_1}^{N/N-s} \leq \left(\frac{c_{\eta_1, \eta_2, \eta_3} + \xi}{\frac{1}{p_1} - \frac{1}{\theta}}\right)^{N/N-s} \leq \left(\frac{C_{\gamma_1} + \xi}{\frac{1}{p_1} - \frac{1}{\theta}}\right)^{N/N-s} < \frac{\alpha_*}{l\alpha_0} \sigma^{s/N-s} \tag{17}$$

for some $l > 1$, choose near 1 and γ_1 is sufficiently large. By using fractional Trudinger-Moser inequality [7] and $\max 1, \eta_1^{-1/p} \|u\|_{\eta_1, p} \leq \|u\|_{W^{s,p} \mathbb{R}^N} \leq \min 1, \eta_1^{-1/p} \|u\|_{\eta_1, p}$, we deduce

$$\sup_{u \in W^{s,p} \mathbb{R}^N, \|u\|_{\eta_1, p} \leq \min 1, \eta_1^{s/N} \int_{\mathbb{R}^N} \Phi_{N,s} \alpha |u|^{N/N-s} dx < +\infty \tag{18}$$

for all $0 \leq \alpha < \alpha_* \leq \alpha_{s,N}^*$. By arguments as Lemma 3.3 [10], we get $\mathcal{J}'_{\eta_1, \eta_2, \eta_3} u = 0$. It is assumed that $u \neq 0$, then it is a weak solution to (1). When $u \equiv 0$, we find solution of (1) in the following way. We assume that

$$\limsup_{n \rightarrow \infty} \int_{y \in \mathbb{R}^N} \int_{B_R y} |u_n x|^p dx = 0, \tag{19}$$

then from Lemma 4, we deduce $u_n \rightarrow 0$ in $L^q \mathbb{R}^N$ for all $q > p = \frac{N}{s}$. Using (17), (19) and by arguments as [9] together with Trudinger-Moser inequality, we get

$$\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * H_1 u_n \right] h_1 u_n u_n dx \rightarrow 0 \text{ and } \int_{\mathbb{R}^N} h_2 u_n u_n dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, it holds

$$\begin{aligned} o 1 = \langle \mathcal{J}'_{\eta_1, \eta_2, \eta_3} u_n, u_n \rangle &= \sum_{i=1}^m \|u_n\|_{p_j, \eta_1}^{p_j} - \eta_2^2 \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * H_1 u_n \right] h_1 u_n u_n dx \\ &\quad - \eta_3 \int_{\mathbb{R}^N} h_2 u_n u_n dx + o_n 1 = \sum_{i=1}^m \|u_n\|_{p_j, \eta_1}^{p_j} + o_n 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence $u_n \rightarrow 0$ in \mathfrak{M} . Therefore, it is obtained as

$$\mathcal{J}_{\eta_1, \eta_2, \eta_3} u_n = \sum_{i=1}^m \frac{1}{p_j} \|u_n\|_{p_j, \eta_1}^{p_j} - \frac{\eta_2^2}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * H u_n \right] H u_n dx - \eta_3 \int_{\mathbb{R}^N} H_2 u_n dx \rightarrow 0$$

as $n \rightarrow \infty$. It is a contradiction with $\lim_{n \rightarrow \infty} \mathcal{J}_{\eta_1, \eta_2, \eta_3} u_n = c_{\eta_1, \eta_2, \eta_3} > 0$. Hence (19) must hold.

Denote by $v_n x := u_n x + y_n$, it comes from (19) that

$$\int_{B_R 0} |v_n|^p dx \geq \delta / 2 > 0 \tag{20}$$

for some $\delta > 0$. By the translation invariance of $\mathcal{J}_{\eta_1, \eta_2, \eta_3}$ and $\mathcal{J}'_{\eta_1, \eta_2, \eta_3}$, we get $\mathcal{J}_{\eta_1, \eta_2, \eta_3} v_n \rightarrow c_{\eta_1, \eta_2, \eta_3}$ and $\mathcal{J}'_{\eta_1, \eta_2, \eta_3} u_n \rightarrow 0$ in \mathfrak{M}^* .

Because $\|v_n\|_{p_j, \eta_1} = \|u_n\|_{p_j, \eta_1}$, for all $j = 1, \dots, m$, then v_n is also bounded in \mathfrak{M} , and then is $v \in \mathfrak{M}$ so that $v_n \rightharpoonup v$ in \mathfrak{M} , $v_n \rightarrow v$ in $L_{loc}^q \mathbb{R}^N$ for all $q \in \left[\frac{N}{s}, +\infty\right)$, $v_n \rightarrow v$ in outside a set measure zero. From (20), we see that $\int_{B_R} |v|^p dx \geq \delta/2 > 0$. So $v \neq 0$. By the same arguments as Lemma 3.3 [10], we get $\mathcal{J}_{\eta_1, \eta_2, \eta_3} v = 0$, $v \neq 0$ in \mathfrak{M} .

4. Conclusion

In this paper, we study the existence of weak solution to a class of fractional anisotropic equation with exponential growth in the sense of Trudinger-Moser. We consider mixed nonlinearities with Choquard reaction. To prove main result, we use Lions's result and Mountain Pass Theorem. The main result of this research can be used to study the existence and multiplicity of weak solutions for the autonomous problem.

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