ISSN: 1859-2171
e-ISSN: 2615-9562 TNU Journal of Science and Technology 208(15): 221 - 227

VỀ SỰ TỒN TẠI NGHIỆM YẾU CỦA BÀI TOÁN KRICHHOFF THỨ VỚI SỐ MŨ TỚI HẠN

Phạm Thị Thủy[1*], Đỗ Thị Mai Hương[2]
[1]*Trường Đại học Sư phạm – ĐH Thái Nguyên*
[2]*Trường Cao đẳng Sư phạm Thái Nguyên*

TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu sự tồn tại nghiệm yếu của bài toán Krichhoff chứa toán tử tích vi phân:

$$\begin{cases} -(a+b\|u\|^2_{X_0})\mathcal{L}_K u &= f(x,u) + \lambda u + \gamma |u|^{2^*_s-2}u \text{ trong } \Omega, \\ u &= 0 \qquad\qquad\qquad\qquad \text{trong } \mathbb{R}^3 \setminus \Omega, \end{cases}$$

Trong đó $\lambda, \gamma > 0$ là các tham số thực và Ω là một miền mở bị chặn trong $\mathbb{R}^3$ với biên $\partial\Omega$ Lipschitz, $s \in (3/4, 1)$, *f* là hàm liên tục thỏa mãn một số điều kiện thích hợp.

Từ khóa: *điều kiện Ambrosetti-Rabinowitz, toán tử Laplace thứ, định lý Moutain Pass, bài toán kiểu Krichhoff, điều kiện Cerami*

Ngày nhận bài: 18/11/2019; Ngày hoàn thiện: 26/11/2019; Ngày đăng: 29/11/2019

ON EXISTENCE O F WEEK SOLUTION TO A FRACTIONAL KIRCHHOFF PROBLEM WITH CRITICAL EXPONENT

Pham Thi Thuy[1], Do Thi Mai Huong[2]
[1]*University of Education – TNU,*
[2]*Thai Nguyen Pedagogical College*

ABSTRACT

In this paper, we consider the following nonlocal problem:

$$\begin{cases} -(a+b\|u\|^2_{X_0})\mathcal{L}_K u &= f(x,u) + \lambda u + \gamma |u|^{2^*_s-2}u \text{ in } \Omega, \\ u &= 0 \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases}$$

where *λ, γ* > 0 are real parameters and Ω is an open bounded subset of $\mathbb{R}^3$ with Lipschitz boundary ∂Ω, *s* ∈ (3/4, 1), and the term *f* is a continuous function satisfying some suitable conditions.

Keywords: *Ambrosetti-Rabinowitz condition, Fractional Laplace equation, Mountain Pass Theorem, Kirchhoff type problem, Cerami condition.*

Received: 18/11/2019; Revised: 26/11/2019; Published: 29/11/2019

* Corresponding author. *Email: p.thuysptn@gmail.com*

1 Introduction and main result

In this paper, we consider the following nonlocal problem:

$$\begin{cases} -(a+b||u||_{X_0}^2) & \mathcal{L}_K u = f(x,u) + \lambda u \\ & +\gamma |u|^{2_s^*-2}u \text{ in } \Omega, \\ u & = 0 \qquad \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases}$$

where λ is a real parameter, γ is a non-negative real parameter, and Ω is an open bounded subset of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$, $s \in (3/4, 1)$, and the term f is a continuous function verifying the conditions stated in the sequel. Moreover, a, b denote two positive real constants and

$$||u||_{X_0}^2 := \int_{\mathbb{R}^6} |u(x)-u(y)|^2 K(x-y)dxdy.$$

The $\mathcal{L}_K$ is the integrodifferential operator which is defined as following:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^3} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^3, \qquad (1.1)$$

where the kernel $K : \mathbb{R}^3 \setminus \{0\} \to (0, +\infty)$ is such that

$$mK \in L^1(\mathbb{R}^3), \quad \text{where } m(x) = \min\{|x|^2, 1\}, \qquad (1.2)$$

and there exists $\theta > 0$ such that

$$K(x) \geq \theta |x|^{-(3+2s)} \qquad (1.3)$$

for any $x \in \mathbb{R}^3 \setminus \{0\}$. A model for K is given by the singular kernel $K(x) = |x|^{-(3+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, that may defined (up to a normalizing constant) by the Riesz potential as follows:

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy$$

for any $x \in \mathbb{R}^3$.

Definition 1. We say that $u \in X_0$ is a weak solution of problem (1.1) if

$$(a+b||u||_{X_0}^2) \int_{\mathbb{R}^3\times\mathbb{R}^3} (u(x)-u(y)) \times (\varphi(x)-\varphi(y))K(x-y)dxdy = \int_\Omega f(x,u(x))\varphi(x)dx + \lambda \int_\Omega u(x)\varphi(x)dx + \gamma \int_\Omega |u(x)|^{2_s^*-2}u(x)\varphi(x)dx$$

for any $\varphi \in X_0$. Here, the space X_0 is defined by

$$X_0 := \{g \in X : g = 0 \text{ in } x \in \mathbb{R}^3 \setminus \Omega\},$$

where the functional space X denotes by the linear space of Lebesgue measurable functions from $\mathbb{R}^3$ to $\mathbb{R}$ such that the restriction of any function g in X to Ω belong to $L^2(\Omega)$ and the map

$$(x,y) \to (g(x)-g(y))\sqrt{K(x-y)}$$

is in $L^2((\mathbb{R}^3 \times \mathbb{R}^3) \setminus (C\Omega \times C\Omega), dxdy)$, $C\Omega := \mathbb{R}^3 \setminus \Omega$.

We denote $F(x,t) := \int_0^t f(x,\tau)d\tau$ and $G(x,t) = f(x,t)t - 4F(x,t)$, for all $(x,t) \in \Omega \times \mathbb{R}$.

We assume that $f \in C(\Omega \times \mathbb{R})$ satisfies following conditions hold:

(f_0) There exists a positive constant C such that

$$|f(x,t)| \leq C(1+|t|^{q-1}), \forall (x,t) \in \Omega \times \mathbb{R}$$

for some $q \in (4, \frac{6}{3-2s})$;

(f_1) $tf(x,t) \geq 0$ in $\Omega \times \mathbb{R}$;

(f_2) $\lim_{|t|\to+\infty} \frac{f(x,t)}{t^3} = +\infty$, uniformly in $x \in \Omega$.

(f_3) There exist $r > 0, \rho \geq 0, \varphi(x) \geq 0$, $\varphi(x) \in L^1(\Omega)$ such that

$$4F(x,t) - f(x,t)t \leq \rho|t|^\sigma + \varphi(x) \text{ for all } |t| \geq r,$$

where $\sigma \in [0, 2)$.

(f_4) There is $\delta > 0$ such that

$$F(x,t) \leq ht^2,$$

for every $x \in \Omega$ and $t \in (-\delta, \delta)$, where $h \neq 0$ is a real number.

The condition (f_3) is larger than the Ambrosetti-Rabinowitz type condition given in [1] played an important role in the application of Mountain Pass Theorem:

(AR) There exists $\mu > 4$ such that $\mu F(x,t) \leq f(x,t)t$ for $|t|$ large enough and $x \in \Omega$ (see [2]).

Our condition (f_3) is also larger than the condition [3]:

(AR-1) $4F(x,t) \leq f(x,t)t$ for $|t|$ large enough and $x \in \Omega$.

Our result is given as follows:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ and $s \in (\frac{3}{4}, 1)$. Let $f \in C(\Omega \times \mathbb{R})$ satisfies the conditions $(f_0) - (f_4)$. Then there exists $\gamma^* > 0$ such that problem (1.1) has at least one non-trivial weak solution for any $\lambda \in \mathbb{R}$ and $\gamma \in (0, \gamma^*)$.*

In order to study problem (1.1), we consider the Euler-Lagrange equation of energy functional $\mathcal{J}_{K,\lambda,\gamma} : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}_{K,\lambda,\gamma}(u) := \frac{a}{2}||u||^2_{X_0} + \frac{b}{4}||u||^4_{X_0} - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx - \frac{\gamma}{2^*_s}\int_\Omega |u(x)|^{2^*_s} dx - \int_\Omega F(x,u(x))dx. \tag{1.4}$$

2 Some preliminary results

Now, we recall some basic results on the spaces X and X_0. In the sequel we set $Q = \mathbb{R}^6 \setminus \mathcal{O}$, where $\mathcal{O} = C\Omega \times C\Omega \subset \mathbb{R}^6$.

The space X is endowed with the norm defined as

$$||g||_X = ||g||_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y)dxdy\Big)^{1/2}. \tag{2.1}$$

It is easily seen that $||.||_X$ is a norm on X (see, for instance, [4] for a proof). Futhermore, X_0 is endowed with norm

$$||g||_{X_0} = \Big(\int_{\mathbb{R}^3\times\mathbb{R}^3} |g(x)-g(y)|^2 K(x-y)dxdy\Big)^{1/2}, \tag{2.2}$$

and $(X_0, ||.||_{X_0})$ is a Hilbert space (see [4], Lemma 7), with scalar product

$$< u, v >_{X_0} = \int_{\mathbb{R}^3\times\mathbb{R}^3} (u(x)-u(y)) \times (v(x)-v(y))K(x-y)dxdy. \tag{2.3}$$

In the following we denote $H^s(\Omega)$ the usual fractional Sobolev space endowed with norm (the so-call *Gagliardo norm*)

$$||g||_{H^s(\Omega)} = ||g||_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{3+2s}} dxdy\Big)^{1/2}. \tag{2.4}$$

We recall that the space X_0 is nonempty (see Lemma 5.2 [5]). Finally, we recall that the eigenvalue problem driven by $-\mathcal{L}_K$, namely

$$\begin{cases} -\mathcal{L}_K u &= \lambda u \text{ in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega. \end{cases} \tag{2.5}$$

We know that (2.5) [6] possesses a divergent sequence of positive eigenvalues

$$\lambda_1 < \lambda_2 < \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots,$$

whose corresponding eigenfunctions will be denoted by e_k, each eigenvalue λ_k has finite multiplicity. By Proposition 9 in [6], we know that $\{e_k\}_{k\in\mathbb{N}}$ can be choosen in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0.

The following result due to Servadei-Valdioci which give the characteristic for embedding from X_0 into $L^\nu(\mathbb{R}^3), \nu \in [1, 2^*_s], 2^*_s = \dfrac{6}{3-2s}$:

Lemma 1. *[7] Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying (1.2)- (1.3). Then, the following assertions hold true:*

a) if Ω is a bounded domain with continuous boundary, then embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^3)$ is compact for any $\nu \in [1, 2_s^*)$;
b) the embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^3)$ is continuous for all $\nu \in [1, 2_s^*]$.

From Lemma 1, we have embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^3)$ is continuous for all $\nu \in [1, 2_s^*]$. Then there exists the best constant

$$S_\nu = \inf_{v \in X_0, v \neq 0} \frac{\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|v(x)-v(y)|^2}{|x-y|^{3+2s}} dxdy}{\left(\int\limits_{\mathbb{R}^3} |v(x)|^\nu dx\right)^{2/\nu}}. \quad (2.6)$$

We have

$$\begin{aligned} &< \mathcal{J}_{K,\lambda}^{'}(u), \varphi >= (a + b||u||^2_{X_0}) \\ &\times \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (u(x)-u(y))(\varphi(x)-\varphi(y)) \\ &\times K(x-y)dxdy \\ &- \gamma \int\limits_\Omega |u(x)|^{2_s^*-2} u(x)\varphi(x)dx \\ &- \int\limits_\Omega f(x,u(x))\varphi(x)dx - \lambda \int\limits_\Omega u(x)\varphi(x)dx. \end{aligned}$$

Certainly, solutions of problems (1.1) is critical point of the energy function $\mathcal{J}_{K,\lambda}$.

3 Proof of Theorem 2

In [8, 9], Cerami introduced the so-called *Cerami condition*, as a weak version of the Palais-Smale assumption. With our notation, it can be written as follows:

Cerami condition. The function $\mathcal{J}_{K,\lambda,\gamma}$ satisfies the *Cerami compactness condition* at level $c \in \mathbb{R}$ if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that $\mathcal{J}_{K,\lambda,\gamma}(u_j) \to c$ and $(1+||u_j||_{X_0}) \sup_{||\varphi||_{X_0}=1} |< \mathcal{J}'_{K,\lambda,\gamma}(u_j), \varphi >| \to 0$, admits a strongly convergent subsequence in X_0.

In order to prove Theorem 2, we need the Mountain Pass Theorem as following:

Theorem 3. *[10] Let $(E, ||.||)$ be a real Banach space. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{J(0), J(u_1)\} \le \alpha < \beta \le \inf_{||u||=\rho} J(u)$$

for some $\alpha < \beta, \rho > 0$ and $u_1 \in E$ with $||u_1|| > \rho$. Let

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = u_1\}.$$

Set $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$. Then $c \ge \beta > 0$ and there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset E$ such that

$$J(u_j) \to c, \text{ and } (1 + ||u_j||)J'(u_j) \to 0.$$

Moreover, if J satisfies the Cerami condition, then c is a critical value of J.

Lemma 2. *[11] Let $\{u_n\}_n$ be a bounded sequence in X_0. Then, up to a subsequence, there exists $u \in X_0$, two Borel regular measures η and ν, **J** denumerable, $x_j \in \overline{\Omega}$, $\nu_j \ge 0, \eta_j \ge 0$ with $\nu_j + \eta_j > 0, j \in \mathbf{J}$ such that $q \in [1, 2_s^*)$,*

$$u_n \to u \text{ weakly in } X_0 \text{ and strong in } L^q(\Omega),$$

$$\int\limits_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dy \overset{*}{\rightharpoonup} d\eta, \ |u_n|^{2_s^*} \overset{*}{\rightharpoonup} d\nu, \quad (3.1)$$

$$d\eta \ge \int\limits_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dy + \sum_{j\in\mathbf{J}} \eta_j \delta_{x_j}, \quad (3.2)$$

$$d\nu = |u|^{p_s^*} + \sum_{j\in\mathbf{J}} \nu_j \delta_{x_j}, \quad (3.3)$$

$$\eta_j \ge S_{2_s^*} \nu_j^{2/2_s^*}, \nu_j := \nu(\{x_j\}), \eta_j := \eta(\{x_j\}). \quad (3.4)$$

Using Lemma 2, we get the following result:

Lemma 3. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a function verifying conditions $(f_0)-(f_4)$. Then for any $M > 0$, there exists $\gamma^* > 0$ such that $\mathcal{J}_{K,\lambda,\gamma}$ satisfies the Cerami condition at level $c \le M$ for any $\lambda \in \mathbb{R}$ and $\gamma \in (0, \gamma^*)$.*

Proof. Fix $M > 0$, we set $\gamma^* = \min\left\{aS_{2_s^*}, B\right\}$, where A is given later, $B = \left[(aS_{2_s^*})^{3/2s}\left(\frac{4s-3}{12(M+A)}\right)^{\frac{2_s^*}{2_s^*-\sigma_*}}\right]^{\mathfrak{a}}$, $\mathfrak{a} = \frac{1}{\frac{3}{2s} - \frac{2_s^*}{2_s^*-\sigma_*}}$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a Cerami sequence in X_0 with level c. We split the proof into two steps. First, we show that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 and that it admits a subsequence strongly convergent in X_0.

Step 1. *The sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.* By normal computing, we have

$$\begin{aligned}&\mathcal{J}_{K,\lambda,\gamma}(u_j)-\frac{1}{4}<\mathcal{J}'_{K,\lambda,\gamma}(u_j),u_j>\\&\geq\frac{a}{4}||u_j||^2_{X_0}-\frac{\lambda}{4}||u_j||^2_{L^2(\Omega)}\\&-\frac{\rho}{4}||u_j||^{\sigma_*}_{L^{\sigma_*}(\Omega)}-C,\end{aligned}\tag{3.5}$$

where $C=\int\limits_{\{x\in\Omega:|u_j(x)|\leq r\}}\Big(F(x,u_j(x))-\frac{1}{4}f(x,u_j(x))u_j(x)\Big)dx+C_1$, $C_1=\frac{\rho|\Omega|}{4}+\frac{1}{4}\int\limits_\Omega\varphi(x)dx$ and $\max\{1,\sigma\}\leq\sigma_*<2$. Since the embedding from $X_0\to L^{\sigma_*}(\Omega)$ is continuous, then there exists the best constant

$$S_{\sigma_*}=\inf_{v\in X_0,v\neq0}\frac{\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v(x)-v(y)|^2}{|x-y|^{3+2s}}dxdy}{\Big(\int\limits_{\mathbb{R}^3}|v(x)|^{\sigma_*}dx\Big)^{2/\sigma_*}}.$$

From (3.5), we obtain

$$\begin{aligned}C_{K,\lambda}||u_j||^2_{X_0}\leq&\frac{\rho S_{\sigma_*}^{-\sigma_*/2}}{4}||u||^{\sigma_*}_{X_0}\\&+\kappa(1+||u_j||_{X_0})+C\end{aligned}$$

for any $j\in\mathbb{N}$, where (see [6], Lemma 16)

$$C_{K,\lambda}=\begin{cases}\frac{a}{4} & \text{if }\frac{\lambda}{a}\leq0\\ \frac{a}{4}-\frac{\lambda}{4\lambda_1} & \text{if }0<\frac{\lambda}{a}<\lambda_1\\ \frac{a}{4}-\frac{\lambda}{4\lambda_{k+1}} & \text{if }\lambda_k\leq\frac{\lambda}{a}<\lambda_{k+1}.\end{cases}$$

Therefore, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

Step 2. The property Cerami compactness condition of $\{u_j\}$. Given $\gamma<\gamma_*$ and $c<M$, let us consider a $(C)_c$ sequence $\{u_j\}_{j\in\mathbb{N}}$ for $J_{K,\lambda,\gamma}$. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 by Step 1 and X_0 is a reflexive space (being Hilbert space, by Lemma 7 in [4]), up to a subsequence, still denote by $\{u_j\}_{j\in\mathbb{N}}$, there exist $u_\infty\in X_0$ such that

$$\begin{aligned}&u_j\to u_\infty\text{ in }L^q(\mathbb{R}^3)\\&u_j\to u_\infty\text{ in }\mathbb{R}^3\end{aligned}\tag{3.6}$$

as $j\to+\infty$ and apply to Lemma A.1 in [12], there exists $l\in L^q(\mathbb{R}^3)$ such that

$$\{|u_\infty(x)|,|u_j(x)|\}\leq l(x)\tag{3.7}$$

for all $x\in\mathbb{R}^3$ and for any $j\in\mathbb{N}$.

Note that $C_{K,\lambda}>0$ and apply Hölder inequality, we get

$$\begin{aligned}&\mathcal{J}_{K,\lambda,\gamma}(u_j)-\frac{1}{4}<\mathcal{J}'_{K,\lambda,\gamma}(u_j),u_j>\\&\geq\gamma\Big(\frac{1}{4}-\frac{1}{2^*_s}\Big)||u_j||^{2^*_s}_{L^{2^*_s}(\Omega)}\\&-\frac{\rho}{4}|\Omega|^{\frac{2^*_s-\sigma_*}{2^*_s}}||u_j||^{\sigma_*}_{L^{2^*_s}(\Omega)}-C.\end{aligned}\tag{3.8}$$

Now, we proceed by some substeps as follows:

Step 2.1. Apply Lemma 2, fix $i_0\in\mathbf{J}$. Then, either $\nu_{i_0}=0$ or

$$\nu_{i_0}\geq\Big(\frac{aS_{2^*_s}}{\gamma}\Big)^{3/2s}.$$

Step 2.2. We claim that $v_{i_0}\geq\Big(\frac{aS_{2^*_s}}{\gamma}\Big)^{3/2s}$ can not occur, hence $\nu_{i_0}=0$. It is enough to show that

$$\int\limits_\Omega d\nu<\Big(\frac{aS_{2^*_s}}{\gamma}\Big)^{3/2s}.\tag{3.9}$$

First of all, we assume that $\int\limits_\Omega d\nu\leq1$. From the assumption of γ, we have $\gamma<aS_{2^*_s}$, it implies $\frac{aS_{2^*_s}}{\gamma}>1$, which immediately get (3.9).

Now, we assume that $\int\limits_\Omega d\nu>1$. Since $\{u_j\}_{j\in\mathbb{N}}$ is a $(PS)_c$ sequence for $J_{K,\lambda,\gamma}$, take $j\to\infty$ in (3.8) and using (3.1), we get

$$\Big(\frac{1}{4}-\frac{1}{2_s^*}\Big)\gamma\int\limits_\Omega d\nu \le c+C+\frac{1}{4}|\Omega|^{\dfrac{2_s^*-\sigma_*}{2_s^*}}\times\Big(\int\limits_\Omega d\nu\Big)^{\sigma_*/2_s^*}$$

$$\le (c+C+\frac{\rho|\Omega|^{\dfrac{2_s^*-\sigma_*}{2_s^*}}}{4})\Big(\int\limits_\Omega d\nu\Big)^{\sigma_*/2_s^*}$$

$$\le (M+A)\Big(\int\limits_\Omega d\nu\Big)^{\sigma_*/2_s^*},$$

thanks to the choice of $c \le M$ and $A = C + \frac{\rho|\Omega|^{\dfrac{2_s^*-\sigma_*}{2_s^*}}}{4}$. It implies that

$$\int\limits_\Omega d\nu \le \Big[\frac{12(M+A)}{\gamma(4s-3)}\Big]^{\dfrac{2_s^*}{2_s^*-\sigma_*}}. \quad (3.10)$$

By the choice γ^*, we have

$$\Big[\frac{12(M+A)}{\gamma(4s-3)}\Big]^{\dfrac{2_s^*}{2_s^*-\sigma_*}} < \Big(\frac{aS_{2_s^*}}{\gamma}\Big)^{3/2s}. \quad (3.11)$$

From (3.10) and (3.11), we get immediately again (3.9). We have been completed the Step 2.2. Hence, we must have $\nu_{i_0}=0$. Since $i_0 \in \mathbf{J}$ is arbitrary, then from (3.1) and (3.3), we get

$$\lim_{j\to\infty}\int\limits_\Omega |u_j(x)|^{2_s^*}dx = \int\limits_\Omega |u_\infty(x)|^{2_s^*}dx. \quad (3.12)$$

By normal computing, we get $u_j \to u_0$ strongly convergent in X_0. □

Lemma 4. *There exists two constants $\rho, \beta > 0$, such that $\mathcal{J}_{K,\lambda,\gamma} \ge \beta$ for every $u \in X_0$ with $||u||_{X_0} = \rho$.*

Proof. By (f_0) and (f_4), there exists $C_2 > 0$ such that

$$F(x,t) \le ht^2 + C_2|t|^q, \text{ for all } (x,t) \in \Omega\times\mathbb{R}. \quad (3.13)$$

Then from (3.13), we have

$$\begin{aligned}\mathcal{J}_{K,\lambda,\gamma}(u) &= \frac{a}{2}||u||^2_{X_0} + \frac{b}{4}||u||^4_{X_0}\\ &-\frac{\lambda}{2}\int\limits_\Omega |u(x)|^2dx - \frac{\gamma}{2_s^*}\int\limits_\Omega |u(x)|^{2_s^*}dx\\ &-\int\limits_\Omega F(x,u(x))\\ &\ge \frac{a}{2}||u||^2_{X_0} + \frac{b}{4}||u||^4_{X_0} - \frac{\lambda}{2}\int\limits_\Omega |u(x)|^2dx\\ &-h\int\limits_\Omega |u(x)|^2dx - C_2\int\limits_\Omega |u(x)|^qdx\\ &-\frac{\gamma}{2_s^*}\int\limits_\Omega |u(x)|^{2_s^*}dx.\end{aligned}$$

Then, we get

$$\begin{aligned}\mathcal{J}_{K,\lambda,\gamma}(u) &\ge C^*_{K,\lambda}||u||^2_{X_0} + \frac{b}{4}||u||^4_{X_0}\\ &-C_2\int\limits_\Omega |u(x)|^qdx - \frac{\gamma}{2_s^*}\int\limits_\Omega |u(x)|^{2_s^*}dx\\ &\ge \frac{b}{4}||u||^4_{X_0} - C_2\int\limits_\Omega |u(x)|^qdx\\ &-\frac{\gamma}{2_s^*}\int\limits_\Omega |u(x)|^{2_s^*}dx. \quad (3.14)\end{aligned}$$

Therefore, from (2.6) and (3.14), we have

$$\begin{aligned}\mathcal{J}_{K,\lambda,\gamma}(u) &\ge \frac{b}{4}||u||^4_{X_0} - C_3||u||^q_{X_0}\\ &-C_4||u||^{2_s^*}_{X_0}\\ &= ||u||^4_{X_0}(\frac{b}{4} - C_3||u||^{q-4}_{X_0} - C_4||u||^{2_s^*-4}_{X_0})\end{aligned}$$

where $C_3 = C_2S_q^{-q/2}, C_4 = \frac{\gamma S_{2_s^*}^{-2_s^*/2}}{2_s^*}$. Since $h(t) = \frac{b}{4} - C_3t^{q-4} - C_4t^{2_s^*-4}$ is continuous on $[0,+\infty)$ and $\lim_{t\to 0^+} h(t) = \frac{b}{4} > 0$, then we can choose $t_0 > 0$ is small such that $h(t) \ge \frac{b}{4} - \varepsilon_1$, for all $\varepsilon_1 > 0$ and all $t \in [0,t_0]$. Special, if we choose $\varepsilon_1 = \frac{b}{8}$, we have $h(t_0) \ge \frac{b}{8}$. It follows that $\mathcal{J}_{K,\lambda,\gamma}(u) \ge \frac{bt_0^4}{8} = \beta$ with $||u||_{X_0} = t_0 = \rho$. We have completed the proof of Lemma 4. □

Lemma 5. *There exists $e \in X_0$ with $||e||_{X_0} > \rho$ such that $\mathcal{J}_{K,\lambda,\gamma}(e) < 0$.*

Proof. By argument as [13]. We have

$$\lim_{j\to\infty} \int_\Omega \frac{F(x, je_1(x))}{j^4} dx = +\infty,$$

where $e_1(x)$ is the first eigenfunction of the operator $-\mathcal{L}_K$ in X_0. We see

$$\frac{\mathcal{J}_{K,\lambda,\gamma}(je_1)}{j^4} = \frac{a}{2j^2}||e_1||^2_{X_0} - \frac{\lambda}{2j^2}\int\limits_\Omega |e_1(x)|^2 dx$$

$$+ \frac{b}{4}||e_1(x)||^4_{X_0}$$

$$- \frac{\gamma j^{2^*_s - 4}}{2^*_s}\int\limits_\Omega |e_1(x)|^{2^*_s} dx - \int\limits_\Omega \frac{F(x, je_1(x))}{j^4} dx$$

$$\to -\infty$$

as $j \to \infty$. Hence, there exists $\nu_0 \in \mathbb{N}$ such that $e := \nu_0 e_1 \in X_0$, it follows that $||e||_{X_0} > \rho$ and $\mathcal{J}_{K,\lambda,\gamma}(e) < 0$. The conclusion is achieved. □

Using Mountain Pas Theorem, Lemma 3, Lemma 4 and Lemma 5, we are easy to get the result of Theorem 2. Note that, we may choose t_0 in Lemma 4 enough small such that

$$\beta = \frac{b}{8}t_0^4 \leq c = M.$$

References

[1] A. Ambrosetti, P. Rabinowitz, "Dual variational methods in critical point theory and applications", *J. Funct. Anal*, vol. 14, pp. 349-381, 1973.

[2] J. Sun, S. Liu, "Nontrivial solution of Kirchhoff type problems", *Appl. Math. Lett*, vol. 25, pp. 500-504, 2012.

[3] J. Jin, X. Wu, "Infinitely many radial solutions for Kirchhoff-type problems in $\mathbb{R}^N$", *J. Math. Anal. Appl*, vol. 369, pp. 564-574, 2012.

[4] R. Servadei, E. Valdinoci, "Mountain Pass solutions for non-local elliptic operators", *J. Math. Anal. Appl*, vol. 389, pp. 887-898, 2012.

[5] R. Servadei, E. Valdinoci, "Lewy-Stampacchia type estimates for variational inequalities driven by (non) local operators", *Rev. Mat. Iberoam*, vol. 29, pp. 1091-1126, 2013.

[6] R. Servadei, E. Valdinoci, "Variational methods for non-local operators of elliptic type", *Discrete Contin. Dyn. Syst*, vol. 33, pp. 2105-2137, 2013.

[7] R. Servadei, E. Valdinoci, "The Brezis-Nirenberg result for the fractional Laplacian", *Trans. Amer. Math. Soc*, vol. 367, pp. 67-102, 2015.

[8] G. Cerami, "An existence criterion for the critical points on unbounded manifolds (in Italian)", *Ist. Lombardo Accad. Sci. Lett. Rend. Sez. A*, vol. 112, pp. 332-336, 1978.

[9] G. Cerami, "On the existence of eigenvalues for a nonlinear boundary value problem (in Italian)", *Ann. Mat. Pura Appl*, vol. 124, pp. 161-179, 1980.

[10] S. Mosconi and M. Squassina, "Nonlocal problems at near critical growth", *Nonlinear Anal*, vol. 136, pp. 84-101, 2016.

[11] M. Schechter, "A variation of the mountain pass lemma and applications", *J. London. Math. Soc*, vol. 44(2), pp. 491-502, 1991.

[12] M. Willem, "Minimax Theorems", Birkhäuser, Basel, 1996.

[13] B. M. Giovanni, F. Tulone, "An existence result for fractional Kirchhoff-type equations", *Journal of Analysis and its Applications*, vol. 35, pp. 181-197, 2016.