ON FIXED POINT THEOREMS FOR MIXED MONOTONE OPE RATORS WITHOUT NORMALITY OF CONE

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ABSTRACT

Mixed monotone operators were introduced by Dajun Guo and V. Lakshmikantham in 1987. Thereafter many authors have investigated these kinds of operators in Banach spaces. In 2009, Z. Zhang- K. Wang proved the fixed point theorems for mixed monotone operators in Banach spaces under the normal cone assumption. In this paper, we establish existence results of fixed point for mixed monotone operators in a real Hausdorff locally convex topological vector space without normality of cone. Our result is an extension of Z. Zhang- K. Wang.

VỀ ĐỊNH LÝ ĐIỂM BẤT ĐỘNG ĐỐI VỚI TOÁN TỬ ĐƠN ĐIỆU HỖN HỢP MẶC DÙ KHÔNG CẦN TÍNH CHUẨN TẮC CỦA NÓN

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TỪ KHÓA

Nón
Nón chuẩn tắc
Tính chất lân cận
Điểm bất động
Toán tử đơn điệu hỗn hợp

TÓM TẮT

Toán tử đơn điệu hỗn hợp được giới thiệu bởi Dajun Guo và V. Lakshmikantham năm 1987. Sau đó nhiều tác giả đã nghiên cứu các loại toán tử này trong không gian Banach. Năm 2009, Z. Zhang- K. Wang chứng minh định lý điểm bất động đối với toán tử đơn điệu hỗn hợp trong không gian Banach dưới giả thiết nón chuẩn tắc. Trong bài báo này, chúng tôi chứng minh định lý điểm bất động đối với toán tử đơn điệu hỗn hợp trong không gian lồi địa phương Hausdorff không có tính chuẩn tắc của nón. Kết quả của chúng tôi là mở rộng kết quả của Z. Zhang- K. Wang.

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1 Introduction and preliminaries

Mixed monotone operators were introduced by Dajun Guo and V. Lakshmikantham in [1] in 1987. Thereafter many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results (see [2]-[6]). In this paper, we establish existence results of fixed point for mixed monotone operators in a real Hausdorff locally convex topological vector space without normality of cone.

Let E always be a real Hausdorff locally convex topological vector spaces with its zero vector θ and P is subset of E. We say that P is a cone in E if

(i) P is closed, nonempty and $P \neq \{\theta\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone P in E, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P, $x \prec y$ if and only if $x \preceq y$ and $x \neq y$. In this paper, we always suppose E be a real Hausdorff locally convex topological vector spaces, P be a cone in E with $\operatorname{int} P \neq \emptyset$ and $\preceq$ is partial ordering with respect to P. If $x_1, x_2 \in E$, the set $[x_1; x_2] = \{x \in E : x_1 \preceq x \preceq x_2\}$ is called the order interval between x_1 and x_2.

Definition 1.1. *Let P be a cone in E. We say that P satisfies the neighborhood property if for any neighborhood U of θ in E, there is neighborhood V of θ in E such that*

$$(V + P) \cap (V - P) \subset U.$$

Remark 1.2. *If P has a closed convex bounded base then P satisfies the neighborhood property (see, Proposition 1.8 in [4]).*

Example 1.3. *Let $E = \mathbb{R}, P = \mathbb{R}_+$. We early can check that P satisfies the neighborhood property.*

Definition 1.4. *Let E is a topological vector space and $\{x_n\}$ be a sequence in E. We say that*

(i) $\{x_n\}$ is converges to $x \in E$, we denoted $\lim\limits_{n\to\infty} x_n = x$, if for each U is neighborhood of x, there exists n_0 such that

$$x_n \in U \text{ for all } n \geq n_0.$$

(ii) $\{x_n\}$ is Cauchy sequence in E if

$$\lim_{n,m\to\infty} (x_n - x_m) = \theta.$$

(iii) E is complete if every Cauchy sequence is converges.

Lemma 1.5. *Suppose that P satisfies the neighborhood property in a real Hausdorff locally convex topological vector space E. Let $\{u_n\}, \{v_n\}, \{w_n\}$ be sequences in E such that $w_n \preceq u_n \preceq v_n$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} w_n = \theta$. Then $\lim\limits_{n\to\infty} u_n = \theta$.*

Proof. Let U be an arbitrary neighborhood of θ in E. Since P satisfies the neighborhood property, there is neighborhood V of θ in E such that

$$(V + P) \cap (V - P) \subset U.$$

Since $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = \theta$, there exists n_0 such that $v_n, w_n \in V$ for all $n \geq n_0$. On the other hand, by $w_n \preceq u_n \preceq v_n$ for all $n \geq 1$, we have $u_n \in (w_n + P) \cap (v_n - P)$ for all $n \geq 1$. Thus $u_n \in (V+P) \cap (V-P)$ for all $n \geq n_0$. Therefore, $u_n \in U$ for all $n \geq n_0$. Hence $\lim_{n\to\infty} u_n = \theta$. □

Definition 1.6. (See [5]) Let P be a cone in normed space E. We say that P is normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq M\|y\|.$$

Proposition 1.7. *Let P be a normal cone in normed space E. Then P satisfies the neighborhood property.*

Proof. Assume that P does not satisfy the neighborhood property. Then there exists $\epsilon > 0$ such that for any $n \geq 1$, we have

$$[B(\theta, \frac{1}{n}) + P] \cap [B(\theta, \frac{1}{n}) - P] \not\subset B(\theta, \epsilon),$$

where $B(\theta, \delta) = \{x \in E : \|x\| < \delta\}$. Thus, for any $n \geq 1$, there exists $u_n \in P, v_n \in B(\theta, \frac{1}{n})$ such that $u_n \preceq v_n$ and $u_n \notin B(\theta, \epsilon)$. Since P is normal cone and $\lim_{n\to\infty} v_n = \theta$, then $\lim_{n\to\infty} u_n = \theta$. Hence $\theta \notin B(\theta, \epsilon)$. This is a contradiction. □

2 Mail results

Definition 2.1. *Let P be a cone in E. We say that $A : P \times P \to P$ is a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., for $u_1, u_2, v_1, v_2 \in P$ with $u_1 \preceq u_2$, $v_2 \preceq v_1$ implies $A(u_1, v_1) \preceq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.*

Theorem 2.2. *Suppose that the cone P satisfy the neighborhood property in a complete real Hausdorff locally convex topological vector space E and $A : P \times P \to P$ be a mixed monotone operator satisfying the following conditions:*

(i) for each $v \in P$, $A(., v) : P \to P$ is concave;

(ii) for any $u \in P$, there exists $N > 0$ such that

$$-N(v_1 - v_2) \preceq A(u, v_1) - A(u, v_2) \text{ for all } v_1, v_2 \in P, v_2 \preceq v_1;$$

(iii) there exists $\bar{v} \in P, \theta \prec \bar{v}$ and $\delta \in (0, 1]$ such that

$$\theta \prec A(\bar{v}, \theta) \preceq \bar{v} \text{ and } \delta A(\bar{v}, \theta) \preceq A(\theta, \bar{v}).$$

Then A has a unique fixed point $u^* \in [\theta, \bar{v}]$ with

$$A(\theta, \bar{v}) \preceq u^* \preceq A(\bar{v}, \theta).$$

Proof. We prove the theorem in three steps:

Step 1. We show that for each $u \in [\theta; \bar{v}]$, $A(u, .)$ has a unique fixed point $T(u)$ on $[A(\theta; \bar{v}); A(\bar{v}, \theta)]$. Indeed, for any $u \in [\theta; \bar{v}]$, there is $N > 0$ such that $A(u, v) + Nv$ is increasing in v by (ii). Put

$$B(u, v) := \frac{A(u, v) + Nv}{N + 1}.$$

Because A is mixed monotone, $B(u,v)$ is increasing in u and v. Moreover, we have $B(u,v)=v$ if and only if $A(u,v)=v$. Let $\{u_n\}$ và $\{v_n\}$ are two sequences as:
$u_0=A(\theta,\bar{v}), v_0=A(\bar{v},\theta)$ and

$$u_{n+1}=B(u,u_n), v_{n+1}=B(u,v_n) \text{ for all } n\geq 0.$$

By $\theta\preceq u\preceq\bar{v}, u_0\preceq A(\theta,\bar{v})\preceq\bar{v}, u_0\preceq v_0$ and $v_0=A(\bar{v},\theta)>\theta$, we have

$$u_0=A(\theta,\bar{v})\preceq A(u,u_0) \text{ and } A(u,v_0)\preceq A(\bar{v},\theta)=v_0.$$

This implies that

$$u_0\preceq\frac{A(u,u_0)+Nu_0}{N+1}=B(u,u_0)=u_1,$$
$$v_1=B(u,v_0)=\frac{A(u,v_0)+Nv_0}{N+1}\preceq v_0.$$

Hence, $u_0\preceq u_1\preceq v_1\preceq v_0$. By induction we have

$$u_n\preceq u_{n+1}\preceq v_{n+1}\preceq v_n \text{ for all } n\geq 0.$$

Thus, we obtain

$$\begin{aligned}\theta\preceq v_n-u_n&=\frac{A(u,v_{n-1})-A(u,u_{n-1})+N(v_{n-1}-u_{n-1})}{N+1}\\&\preceq\frac{N}{N+1}(v_{n-1}-u_{n-1}) \text{ for all } n\geq 1.\end{aligned}$$

Therefore

$$\theta\preceq v_n-u_n\preceq\Big(\frac{N}{N+1}\Big)^n(v_0-u_0) \text{ for all } n\geq 1.$$

By Lemma 1.5, we get

$$\lim_{n\to\infty}(v_n-u_n)=\theta.$$

On the other hand, for $m>n$ we have

$$\theta\preceq u_m-u_n\preceq v_m-u_n\preceq v_n-u_n.$$

Since $\lim_{n\to\infty}(v_n-u_n)=\theta$ and by Lemma 1.5, we get

$$\lim_{m,n\to\infty}(u_m-u_n)=\theta.$$

Thus, $\{u_n\}$ is a Cauchy sequence in E. By E is complete, there is $T(u)\in E$ such that

$$\lim_{n\to\infty}u_n=T(u)=\lim_{n\to\infty}v_n.$$

For each $n\geq 0$, we have

$$u_n\preceq u_{n+k}\preceq v_{n+k}\preceq v_n \text{ for all } k\geq 1$$

Letting $k\to\infty$, we have

$$u_n\preceq T(u)\preceq v_n \text{ for all } n\geq 1.$$

Since B is increasing, we have

$$u_{n+1}=B(u,u_n)\preceq B(u,T(u))\preceq B(u,v_n)=v_{n+1} \text{ for all } n\geq 1.$$

By Lemma 1.5, we get

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = B(u, T(u)).$$

This implies that $T(u) = B(u, T(u))$. Thus, $T(u) \in [A(\theta, \bar{v}); A(\bar{v}, \theta)]$ is a fixed point of $B(u, .)$. If there $x \in [A(\theta, \bar{v}); A(\bar{v}, \theta)]$ with $B(u, x) = x$ then

$$u_n \preceq x \preceq v_n \text{ for all } n \geq 1.$$

Therefore, $x = T(u)$. Thus, $T(u) \in [A(\theta, \bar{v}); A(\bar{v}, \theta)]$ is a unique fixed point of $B(u, .)$. Hence, $T(u) \in [A(\theta, \bar{v}); A(\bar{v}, \theta)]$ is a unique fixed point of $A(u, .)$.

Step 2. We show that

(1) $T(.)$ is increasing;

(2) $[(1-\delta)t + \delta]T(u) \preceq T(tu)$ for all $t \in [0, 1]$.

Let $u, u' \in [\theta, \bar{v}], u \preceq u'$. From **Step 1**, we choose $u_0 = u'_0 = A(\theta, \bar{v})$ and by $B(u, v)$ is increasing in both variables, we have

$$u_1 = B(u, u_0) \preceq B(u', u'_0) = u'_1.$$

By induction, we get $u_n \preceq u'_n$ for all n. Taking the limit, we have

$$T(u) \preceq T(u').$$

Thus, T is increasing. On the other hand, for $t \in [0, 1]$, by A is concave in the first variable and A is mixed monotone, we have

$$\begin{aligned}
[(1-\delta)t + \delta]T(u) &= tT(u) + (1-t)\delta T(u) \\
&\preceq tA(u, T(u)) + (1-t)A(\theta, T(tu)) \\
&\preceq tA(u, T(tu)) + (1-t)A(\theta, T(tu)) \\
&\preceq A(tu, T(tu)) \\
&= T(tu).
\end{aligned}$$

Step 3. We show that T has a unique fixed point on $[\theta; \bar{v}]$. Consider two sequences $\{u_n\}$ and $\{v_n\}$ by: $u_0 = \theta, v_0 = \bar{v}$ and

$$u_{n+1} = T(u_n), v_{n+1} = T(v_n) \text{ for all } n \geq 0.$$

From $T(u) \in [A(\theta; \bar{v}), A(\bar{v}, \theta)] \subset [\theta; \bar{v}]$ for all $u \in [\theta; \bar{v}]$ we have $u_0 \preceq u_1, v_1 \preceq v_0$. This implies

$$u_n \preceq u_{n+1}, v_{n+1} \preceq v_n \text{ for all } n \geq 0.$$

By $u_1, v_1 \in [A(\theta, \bar{v}); A(\bar{v}, \theta)]$ then $\delta v_1 \preceq u_1$. For each n, we put

$$t_n := \sup\{t > 0 : tv_n \preceq u_n\}.$$

Then $t_n \geq \delta$ for all n and $\{t_n\}$ is increasing sequence. Since **Step 2**, for any $n \geq 0$ we have

$$[(1-\delta)t_n + \delta]v_{n+1} = [(1-\delta)t_n + \delta]T(v_n) \preceq T(t_n v_n) \preceq T(u_n) = u_{n+1}.$$

Thus,

$$t_{n+1} \geq (1-\delta)t_n + \delta \text{ for all } n \geq 1.$$

Hence, $\lim_{n\to\infty} t_n = 1$. Moreover, for $m > n$, we have

$$\theta \preceq u_m - u_n \preceq v_m - u_n \preceq v_n - u_n \preceq v_n - t_n v_n \leq (1 - t_n)v_0 \text{ for all } n \geq 1.$$

By Lemma 1.5, we get $\{u_n\}$ is a Cauchy sequence in E. Therefore, there is $u^* \in E$ such that $\lim_{n\to\infty} u_n = u^*$. Now, by

$$u_n \preceq v_n \preceq \frac{1}{t_n} u_n \text{ for all } n \geq 1$$

we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = u^*.$$

By **Step 1**, u^* is a fixed point of T. The proof of the uniqueness is the same as above in **Step 1**. Hence, $u^* \in [\theta; \bar{v}]$ is a fixed point of A with

$$A(\theta, \bar{v}) \preceq u^* \preceq A(\bar{v}, \theta).$$

Assume that v^* is a fixed point of A on $[\theta; \bar{v}]$. By **Step 1**, $T(v^*)$ is a fixed point of $A(v^*, .)$. Since the uniqueness of the fixed point of T, we get $u^* = v^*$. □

Remark. When E is a Banach space and P is a normal cone in E then Theorem 2.2 becomes Theorem 2.1 in [7].

3. Conclusion

In this paper, we establish existence results of fixed point for mixed monotone operators in a real Hausdorff locally convex topological vector space without normality of cone. Our result is an extension of Z. Zhang- K. Wang [7].

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