A NOTE ON UNIQUENESS PROBLEM FOR HOLOMORPHIC CURVES WITH HYPERSURFACES

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ABSTRACT

Recently, several uniqueness theorems for holomorphic curves on an annulus have been published. For example, in 2013, H. T. Phuong and T. H. Minh given two uniqueness results with hyperplanes in general position. In 2021, H. T. Phuong and L. Vilaisavanh proved some uniqueness theorems for the holomorphic mappings on an annulus sharing hypersurfaces in general position or hypersurfaces in general position for Veronese embedding. In this paper, we consider the same problem in the case of hypersurfaces in general position by using the second main theorem for holomorphic curves on an annulus with hypersurfaces. The main result of the paper is Theorem 1, which gives us an algebraic condition so that two holomorphic curves on an annulus are equal. The main technique in the paper is based on a form of the second fundamental theorem for holomorphic curve on an annulus with target being hypersurfaces and some other techniques in Nevanlinna-Cartan theory.

MỘT CHÚ Ý VỀ VẤN ĐỀ DUY NHẤT CHO ĐƯỜNG CONG CHỈNH HÌNH TRÊN HÌNH VÀNH KHUYÊN

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Đường cong chỉnh hình
Lý thuyết Nevanlinna-Cartan
Định lý duy nhất
Vấn đề duy nhất
Hình vành khuyên

TÓM TẮT

Thời gian gần đây, một số kết quả nghiên cứu về định lý duy nhất cho đường cong chỉnh hình trên hình vành khuyên đã được công bố. Chẳng hạn, năm 2013, H. T. Phương và T. H. Minh công bố hai dạng định lý duy nhất với mục tiêu là các siêu phẳng ở vị trí tổng quát. Năm 2021, H. T. Phương và L. Vilaisavanh công bố các kết quả cho trường hợp các siêu mặt ở vị trí tổng quát đối với phép nhúng Veronese. Trong bài báo này, chúng tôi sẽ nghiên cứu vấn đề tương tự cho trường hợp các siêu mặt ở vị trí tổng quát bằng việc sử dụng một dạng định lý cơ bản thứ hai cho trường hợp mục tiêu là các siêu mặt. Kết quả chính của bài báo là Định lý 1, cho chúng ta một điều kiện đại số để hai đường cong chỉnh hình trên một hình khuyên bằng nhau. Kỹ thuật chính trong bài báo dựa trên một dạng của định lý cơ bản thứ hai cho đường cong chỉnh hình trên hình khuyên với mục tiêu là các siêu mặt và một số kỹ thuật khác trong lý thuyết Nevanlinna-Cartan.

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1. Introduction

The uniqueness problem for holomorphic curves is attracted by many authors and there have been some results published. In 1975, H. Fujimoto (see [1]) proved a uniqueness theorem for meromorphic mapping with hyperplanes in general position, which is a improvement of Nevanlinna's Five-Value Theorem to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Since that time, this problem has been studied intensively. For examples: H. Fujimoto (see [2]), Y. Chen and Q. Yan ([3]), G. Dethloff and T. V. Tan ([4]), H. T. Phuong and T. H. Minh ([5]), H. T. Phuong and L. Vilaisavanh ([6]), and more. In this paper, we give a uniqueness result for algebraically non-degenerate holomorphic curves on an annulus sharing sufficiently many hypersurfaces. First of all, we introduce some notations.

Let $R_0 > 1$ be a fixed positive real number or $+\infty$, set

$$\Delta = \big\{z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0\big\},$$

be an annulus in $\mathbb{C}$.

Let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, here $\mathbf{f} = (f_0, \ldots, f_n)$ is a reduced representative of f, namely $f_0, \ldots, f_n$ are holomorphic functions and without common zeros on Δ. For $1 < r < R_0$, characteristic function $T_f(r)$ of f is defined by

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f. And we set

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & \text{if } R_0 = +\infty \\ O(\log \dfrac{1}{R_0 - r} + \log T_f(r)) & \text{if } R_0 < +\infty. \end{cases}$$

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q is the homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ of degree d defining D. Assume

$$Q(z_0, \ldots, z_n) = \sum_{k=0}^{n_d} a_k z_0^{i_{k0}} \ldots z_n^{i_{kn}},$$

where $n_d = \binom{n+d}{n} - 1$ and $i_{k0} + \cdots + i_{kn} = d$ for $k = 0, \ldots, n_d$, then we denote by $\mathbf{a} = (a_0, \ldots, a_{n_d})$ the vector associated with D (or with Q). And we set

$$\overline{E}_f(D) := \{z \in \mathbb{C} \ |Q(f)(z) = 0 \ \text{ ignoring multiplicity } \}.$$

Now let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of arbitrary hypersurfaces and Q_j be the homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$ of degree d_j defining D_j for $j = 1, \ldots, q$. Let $d_{\mathcal{D}}$ is the least common multiple of the $d_j, j = 1, \ldots, q$ and denote

$$n_{\mathcal{D}} = \binom{n + d_{\mathcal{D}}}{n} - 1.$$

We set $Q_j^* = Q_j^{d_{\mathcal{D}}/d_j}$, $j = 1, \ldots, q$, and let $\mathbf{a}_j^*$ is the vector associated with Q_j^*. We recall that the collection of hypersurfaces $\mathcal{D}$ is said to be in *general position for Veronese embedding* if $q > n_{\mathcal{D}}$ and for any distinct $i_1, \ldots, i_{n_{\mathcal{D}}+1} \in \{1, \ldots, q\}$, the vectors $\mathbf{a}_{i_1}^*, \ldots, \mathbf{a}_{i_{n_{\mathcal{D}}+1}}^*$ are linearly independent. The collection $\mathcal{D}$ is said to be *in general position* if $q > n$ and for any distinct $i_1, \ldots, i_{n+1} \in \{1, \ldots, q\}$, $\bigcap_{k=1}^{n+1} \operatorname{supp}(D_{i_k}) = \emptyset$. And we denote

$$\overline{E}_f(\mathcal{D}) := \bigcup_{D \in \mathcal{D}} \overline{E}_f(D).$$

In 2021, H. T. Phuong and L. Vilaisavanh ([6]) proved:

Theorem A. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}^2/d_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Our idea here is to consider this problem in the case of hypersurfaces in general position. Out results are stated as following:

Theorem 1 (Main theorem). *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. If there exist a collection $\mathcal{D} = \{D_1, \ldots, D_q\}$ of $q > n_{\mathcal{D}} + 1 + 2nn_{\mathcal{D}}/d_{\mathcal{D}}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

We know that Theorem 1 gives some uniqueness conditions for algebraically non-degenerate holomorphic maps on an annulus. Note that, the number of necessary hypersurfaces in our result is less than in Theorem A.

2. Some preparations

In this section, we introduce some notations and recall some results in Nevanlinna theory for meromorphic functions and holomorphic curves on annulus, which are necessary for proof of the our main result.

For any real number r such that $1 < r < R_0$, we denote

$$\Delta_{1,r} = \Big\{z \in \mathbb{C} : \frac{1}{r} < |z| \leqslant 1\Big\},\ \Delta_{2,r} = \Big\{z \in \mathbb{C} : 1 < |z| < r\Big\},$$
$$\Delta_r = \Big\{z \in \mathbb{C} : \frac{1}{r} < |z| < r\Big\}.$$

Let f be a meromorphic function on Δ, $c \in \mathbb{C}$, we denote

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta; \quad m(r, \frac{1}{f - c}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - c|} d\theta.$$

Let

$$m_0(r, f) = m(r, f) + m(r^{-1}, f),$$

and

$$m_0(r, \frac{1}{f-c}) = m(r, \frac{1}{f-c}) + m(r^{-1}, \frac{1}{f-c}).$$

Denote by $n_1\left(r, \dfrac{1}{f-c}\right)$ the number of zeros of $f - c$ in $\Delta_{1,r}$, $n_2\left(r, \dfrac{1}{f-c}\right)$ the number of zeros of $f - c$ in $\Delta_{2,r}$, $n_1(r, \infty)$ the number of poles in $\Delta_{1,r}$ and $n_2(r, \infty)$ the number of poles of f in $\Delta_{2,r}$. Put

$$N_1\left(r, \frac{1}{f-c}\right) = \int_{1/r}^{1} \frac{n_1(t, \frac{1}{f-c})}{t} dt, \quad N_2\left(r, \frac{1}{f-c}\right) = \int_{1}^{r} \frac{n_2(t, \frac{1}{f-c})}{t} dt,$$

and

$$N_1(r, f) = N_1(r, \infty) = \int_{1/r}^{1} \frac{n_1(t, \infty)}{t} dt, \quad N_2(r, f) = N_2(r, \infty) = \int_{1}^{r} \frac{n_2(t, \infty)}{t} dt.$$

Set

$$N_0\left(r, \frac{1}{f-c}\right) = N_1\left(r, \frac{1}{f-c}\right) + N_2\left(r, \frac{1}{f-c}\right)$$
$$N_0(r, f) = N_1(r, f) + N_2(r, f).$$

The function

$$T_0(r, f) = m_0(r, f) + N_0(r, f) - 2m(1, f)$$

is called the *Nevanlinna characteristic* of f.

Lemma 2.1. ([7]) *Let f be a nonconstant meromorphic function on Δ. Then for any $r \in (1, R_0)$, we have*

$$N_0\left(r, \frac{1}{f}\right) - N_0(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log |f(r^{-1}e^{i\theta})| d\theta - \frac{1}{\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

Let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve on Δ, where $f_0, \ldots, f_n$ be holomorphic functions and without common zeros on Δ. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and let Q is the homogeneous polynomial of degree d defining D. Let $n_{1,f}(r, D)$ be the number of zeros of $Q(f)$ in $\Delta_{1,r}$, $n_{2,f}(r, D)$ be the number of zeros of $Q(f)$ in $\Delta_{2,r}$, counting multiplicity. The integrated counting functions are defined by

$$N_{1,f}(r, D) = \int_{r^{-1}}^{1} \frac{n_{1,f}(t, D)}{t} dt, \quad N_{2,f}(r, D) = \int_{1}^{r} \frac{n_{2,f}(t, D)}{t} dt,$$

and we set

$$N_f(r, D) = N_{1,f}(r, D) + N_{2,f}(r, D).$$

Let α be a positive integer, we denote by $n^{\alpha}_{1,f}(r,D)$ the number of zeros with multiplicity truncated by α of $Q(f)$ in $\Delta_{1,r}$, $n^{\alpha}_{2,f}(r,D)$ be the number of zeros of $Q(f)$ in $\Delta_{2,r}$ where any zero is counted with multiplicity if its multiplicity is less than or equal to α, and α times otherwise. The integrated truncated counting functions are defined by

$$N^{\alpha}_{1,f}(r,D)=\int\limits_{r^{-1}}^{1}\frac{n^{\alpha}_{1,f}(t,D)}{t}dt,\quad N^{\alpha}_{2,f}(r,D)=\int\limits_{1}^{r}\frac{n^{\alpha}_{2,f}(t,D)}{t}dt,$$

and we set

$$N^{\alpha}_{f}(r,D)=N^{\alpha}_{1,f}(r,D)+N^{\alpha}_{2,f}(r,D).$$

Lemma 2.2. ([6]) *Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and $f=(f_0:\cdots:f_n):\Delta\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained D. Then we have for any $1<r<R_0$,*

$$m_f(r,D)+N_f(r,D)=dT_f(r)+O(1).$$

In 2023, H. T. Phuong and I. Padaphet proved

Lemma 2.3. *Let $f:\Delta\to\mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1\leqslant j\leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d is the least common multiple of the d_j and set $M=\binom{n+d}{n}-1$. Then for any $1<r<R_0$ and $q\geqslant M+1$, we have*

$$\|\quad (q-M-1)T_f(r)\leqslant\sum_{j=1}^{q}\frac{1}{d_j}N^{M}_{f}(r,D_j)+O_f(r).$$

3. Proof of Theorem 1

Assume for the sake contradiction that $f\not\equiv g$. Let $(f_0,\ldots,f_n)$ and $(g_0,\ldots,g_n)$ are reduced representations of f and g, respectively. Let $Q_j, 1\leqslant j\leqslant q$, be the homogeneous polynomials in $\mathbb{C}[z_0,\ldots,z_n]$ of degree d_j defining D_j. Of course we may assume that $q\geqslant n_{\mathcal{D}}+1$. For any $j=1,2,\ldots,q$, we set $Q^*_j=Q_j^{d_{\mathcal{D}}/d_j}$ so that $Q^*_1,\ldots,Q^*_q$ have a same degree of $d_{\mathcal{D}}$. Let D^*_j is hypersurface defining by Q^*_j for $j=1,\ldots,q$. Applying Lemma 2.3 to holomorphic map $f:\Delta\to\mathbb{P}^n(\mathbb{C})$ and the polynomials $D^*_j,\ j=1,\ldots,q$, we have

$$\begin{aligned}\|\quad (q-n_{\mathcal{D}}-1)T_f(r)&\leqslant\frac{1}{d_{\mathcal{D}}}\sum_{j=1}^{q}N^{n_{\mathcal{D}}}_{f}(r,D^*_j)+O_f(r)\\&\leqslant\frac{n_{\mathcal{D}}}{d_{\mathcal{D}}}\sum_{j=1}^{q}N^{1}_{f}(r,D^*_j)+O_f(r).\end{aligned}\tag{1}$$

And similarly for g we get

$$\| \quad (q-n_{\mathcal{D}}-1)T_g(r) \leqslant \frac{n_{\mathcal{D}}}{d_{\mathcal{D}}}\sum_{j=1}^{q} N_g^1(r,D_j^*)+O_g(r). \tag{2}$$

Summing up both sides of (1) and (2), we get

$$\begin{aligned}\| \quad & \frac{d_{\mathcal{D}}}{n_{\mathcal{D}}}(q-n_{\mathcal{D}}-1)(T_f(r)+T_g(r))-(O_f(r)+O_g(r)) \\ & \leqslant \sum_{j=1}^{q}(N_f^1(r,D_j^*)+N_g^1(r,D_j^*)).\end{aligned} \tag{3}$$

Since $f \neq g$, so there exist two numbers $\alpha, \beta \in \{0,\dots,n\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. We set

$$\Phi = f_\alpha g_\beta - f_\beta g_\alpha.$$

Now for any $z \in \Delta$, easy to see that

$$\log|\Phi(z)| \leqslant \log\|f(z)\| + \log\|g(z)\| + \log 2.$$

So from Lemma 2.1 we have for any $r : 1 < r < R_0$,

$$\begin{aligned} N_0\Big(r,\frac{1}{\Phi}\Big) &= \frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(r^{-1}e^{i\theta})|d\theta \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log\|f(r^{-1}e^{i\theta})\|d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\log\|g(re^{i\theta})\|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log\|g(r^{-1}e^{i\theta})\|d\theta + O(1). \\ &= T_f(r)+T_g(r)+O(1). \end{aligned} \tag{4}$$

Now for any $j \in \{1,\dots,q\}$, let $z_0 \in \Delta$ is a zero of $Q_j^*(f)$, from definition of Q_j^*, we get z_0 is a zero of $Q_j(f)$, so $z_0 \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. This implies that $g(z_0) = f(z_0)$, and hence

$$f_\alpha(z_0)g_\beta(z_0) = f_\beta(z_0)g_\alpha(z_0),$$

namely z_0 is a zero of Φ. This implies that

$$\nu_\Phi(z_0) \geqslant \min\{1, \nu_{Q_j^*(f)}(z_0)\}. \tag{5}$$

Note from hypothesis that $\mathcal{D}$ are in general position, there exist at most n hypersurfaces D_j^* such that $Q_j^*(f)(z_0) = 0$. Then from (5), we have

$$\sum_{j=1}^{q} N_f^1(r,D_j^*) \leqslant nN_0\Big(r,\frac{1}{\Phi}\Big).$$

Combining the above inequality with (4), we have

$$\sum_{j=1}^{q} N_f^1(r,D_j^*) \leqslant n(T_f(r)+T_g(r))+O(1).$$

Similarly for g we have

$$\sum_{j=1}^{q} N_g^1(r, D_j^*) \leqslant n(T_f(r) + T_g(r)) + O(1).$$

Summing up both sides of two above inequalities, we get

$$\sum_{j=1}^{q} (N_f^1(r, D_j^*) + N_g^1(r, D_j^*)) \leqslant 2n(T_f(r) + T_g(r)) + O(1). \tag{6}$$

We now combine (6) and (3), we have

$$\| \quad \frac{d_{\mathcal{D}}}{n_{\mathcal{D}}}(q - n_{\mathcal{D}} - 1)(T_f(r) + T_g(r)) - (O_f(r) + O_g(r)) \leqslant 2n(T_f(r) + T_g(r)) + O(1).$$

This implies that

$$\| \quad (q - n_{\mathcal{D}} - 1 - \frac{2nn_{\mathcal{D}}}{d_{\mathcal{D}}}) \leqslant \frac{O_f(r) + O_g(r) + O(1)}{T_f(r) + T_g(r)}.$$

Letting $r \to \infty$ we have

$$q - n_{\mathcal{D}} - 1 - \frac{2nn_{\mathcal{D}}}{d_{\mathcal{D}}} \leqslant 0,$$

this is a contradiction with $q > n_{\mathcal{D}} + 1 + \frac{2nn_{\mathcal{D}}}{d_{\mathcal{D}}}$. Hence $f = g$.

4. Conclusion

In this paper, we have stated and proved a new result about uniqueness theorem for holomorphic curves from an annulus into a complex projective space in the case of hypersurfaces in general position. Note that, when the hypersurfaces in Theorem 1 are hyperplanes then $n_{\mathcal{D}} = n$, $d = 1$ and $q > n + 1 + 2n^2$, so Theorem 1 will get back the previously result for the hyperplanes.

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