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IDEMPOTENT TRIANGULAR MATRICES OVER COMMUTATIVE SEMIRINGS

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ABSTRACT

The structure of idempotent triangular matrices with elements on the main diagonal being 0 or 1 over a commutative ring has been fully described by Xin Hou (2021). These results have also been generalized by Stephen E. Wright (2022) when studying the structure of idempotent triangular matrices over the general rings. Furthermore, Stephen E. Wright (2022) has provided formulas for calculating the number of matrices of this type within finite rings. In this paper, we investigate the characteristic properties of idempotent triangular matrices over commutative semirings and describe the structure of such matrices in cases where the entries on their main diagonal are pairwise orthogonal idempotent elements. Simultaneously, we proceed to compute the number of idempotent triangular matrices with entries on the main diagonal being 0 or 1 when the corresponding semirings are commutative, additively idempotent, and have a finite number of elements.

MA TRẬN TAM GIÁC LŨY ĐẲNG TRÊN NỬA VÀNH GIAO HOÁN

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TỪ KHÓA

Vành
Nửa vành
Ma trận tam giác
Ma trận lũy đẳng
Đường chéo chính

TÓM TẮT

Cấu trúc của các ma trận tam giác lũy đẳng có các phần tử trên đường chéo chính là 0 hoặc 1 trên vành giao hoán đã được Xin Hou (2021) mô tả đầy đủ. Các kết quả này cũng đã được tổng quát hóa bởi Stephen E. Wright (2022) khi nghiên cứu cấu trúc của các ma trận tam giác lũy đẳng trên vành tổng quát. Hơn nữa, Stephen E. Wright (2022) đã cung cấp các công thức tính số ma trận dạng này đối với các vành hữu hạn. Trong bài báo này, chúng tôi khảo sát các tính chất đặc trưng của các ma trận tam giác lũy đẳng trên nửa vành giao hoán và mô tả cấu trúc của các ma trận dạng này trong trường hợp các phần tử trên đường chéo chính của chúng là các phần tử lũy đẳng trực giao từng đôi một. Đồng thời chúng tôi tiến hành tính toán số các ma trận tam giác lũy đẳng có các phần tử trên đường chéo chính là 0 hoặc 1 khi các nửa vành tương ứng là giao hoán, lũy đẳng cộng và có hữu hạn phần tử.

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1. Introduction

Studying the structure of idempotent triangular matrices over rings plays a crucial role in algebra, the structure of idempotent triangular matrices over rings has been fully described in [1] and [2]. In [1], Xin Hou provided a formula for calculating the number of idempotent triangular matrices over a commutative ring with elements 0 or 1 on the main diagonal. The results of Xin Hou were generalized by Stephen E. Wright (see [2]) for idempotent triangular matrices over both commutative and noncommutative rings.

In recent times, along with the strong development of the semiring theory [3], the study of the structure of idempotent matrices on specific classes of semirings has also gained attention, for example, the semiring of non-negative real numbers, boolean algebra, chain semiring, tropical semiring and other semirings (see [4] - [8]). However, there are not many results on the structure of idempotent triangular matrices over general semirings. In this article, we aim to describe the structure of idempotent triangular matrices on commutative semirings, we will investigate idempotent triangular matrices with elements on the main diagonal being 0 or 1, and calculating the number of such matrices on finite semirings that are commutative and additively idempotent.

2. Preliminaries

A *Semiring* [3] is a set *R* equipped two binary operations addition $(+)$ and multiplication $(.)$ such that $(R,+)$ is a commutative monoid with identity element 0, $(R,.)$ is a monoid with identity element 1, multiplication distributes on both sides with addition and $0.s = s.0 = 0, \forall s \in R$.

A semiring R is *commutative* if $a.b = b.a, \forall a,b \in R$; R is said to be *zerosumfree* if $a+b=0 \Rightarrow a=b=0, \forall a,b \in R$; *R* is *additively idempotent* if $a+a=a, \forall a \in R$. Notice that if *R* is additively idempotent then it is zerosumfree. Indeed, if $x,y \in R$ such that $x+y=0$ then $x = x+0 = x+x+y = x+y = 0$ and therefore $y=0$. Let *R* be a semiring that is commutative and additively idempotent, we equip *R* with a binary relation $"\ll"$ (see [7]) as follows: Let $r,t \in R.$ If $r+t=t$ then we write $r \ll t$. Then, $"\ll"$ is a partial ordering. Note that if $a \ll b$ then we can write $b \gg a$ for all $a,b \in R$, if $a \ll b$ and $a \neq b$ then we can write $a \prec b$ or $b \succ a$. It is easy to see that $a \gg b, \forall a \in R$ if and only if $b=0$. Let *R* be a semiring, an element $a \in R$ is *additively invertible* if there exists an element $b \in R$ such that $a+b=0$, the element *b* is denoted by $-a$. The set of all additively invertible elements in *R* is denoted by *V(R)*.

In this paper, the set of all $m \times n$ matrices over semiring R is denoted by $M_{m\times n}(R)$ and $M_n(R)$ if $m=n$; the (i,j) entry of a matrix $A_{m\times n} \in M_{m\times n}(R)$ is denoted by a_{ij} or $a_{i,j}$. Recall [8] that a matrix $A \in M_n(R)$ is called *invertible* if there exists a matrix $B \in M_n(R)$ such that $A.B = B.A = I_n$, the matrix *B* is called *inverse* of *A* and denoted by A^{-1}. We denote by $GL_n(R)$ the set of all invertible matrices in $M_n(R)$. A matrix $A \in M_n(R)$ is called *idempotent* if $A^2 = A$. A matrix $B = (b_{ij}) \in M_n(R)$ is called *(upper) triangular matrix* if $b_{ij} = 0, \forall\ 1 \leq j < i \leq n$, the set of all triangular matrices in $M_n(R)$ is denoted by $TM_n(R)$; let $A,B \in M_{m\times n}(R)$, if there exist $P \in GL_m(R), Q \in GL_n(R)$ such that $A = PBQ$ then we say that *A* and *B* are *equivalent*; for any matrix $A = (a_{ij}) \in M_n(R)$, we denote by $A_1 = (a_{11}); A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; ...; A_n = A$ the *principal submatrices* of *A*.

Proposition 2.1 ([8, Lemma 2.1]). Let R be a semiring, if $s \in V(R)$ then

$rs, sr \in V(R), \forall r \in R$.

Proposition 2.2 ([8, Lemma 2.2]). If $A = (a_{ij}) \in M_n(R)$ is invertible on the commutative semiring *R* then $a_{ik}a_{jk}, a_{ki}a_{kj} \in V(R)$ for all $i, j, k \in \{1, ..., n\}$ with $i \neq j$.

3. Main results

In this section, we establish several characteristic properties of idempotent triangular matrices over semirings which are commutative and additively idempotent, and calculate the number of such matrices in case of finite semirings.

Proposition 3.1. Let *R* be a semiring, $A \in M_k(R), D \in M_p(R), B \in M_{k\times p}(R)$. If the block matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is idempotent then *A* and *D* are also idempotent matrices.

Proof. We have $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^2 = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A^2 & AB+BD \\ 0 & D^2 \end{pmatrix}$, since $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is idempotent, hence, $A^2 = A$ and $D^2 = D$. So, *A* and *D* are idempotent matrices. □

From Proposition 3.1, we easily obtain the following results:

Corollary 3.2. Let *R* be a commutative semiring and $A \in TM_n(R)$. Then A is idempotent if and only if all principal submatrices of *A* are idempotent.

Remark 3.3. If $A = (a_{ij}) \in TM_n(R)$ is idempotent then $a_{ii}^2 = a_{ii}, \forall i = 1, ..., n$.

Proposition 3.4. Let *R* be a commutative semiring and $A = (a_{ij}) \in TM_n(R)$ is an idempotent matrix. If $a_{ii} = 0, \forall i = 1, ..., n$ then $A = (0)$.

Theorem 3.5. Let *R* be a commutative semiring. Then, the following statements are equivalent:

i) For every idempotent triangular matrix $A \in TM_n(R)$ whose main diagonal has *k* entries equal to 1, and the rest equal to 0 ($0 < k < n$), there exist invertible matrices $U, V \in GL_n(R)$ such that $VAU = \begin{pmatrix} I_k & 0 \\ 0 & B \end{pmatrix}$, where all entries of the main diagonal of matrix B are equal to 0;

ii) *R* is a ring.

Proof. $i \Rightarrow ii$: For any $a \in R$, the matrix $A = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ satisfies $A^2 = A$. Hence, A is idempotent. Thus, there exist invertible matrices $U, V \in GL_2(R)$ such that $VAU = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This implies that $A = V^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$. Suppose that $U^{-1} = \begin{pmatrix} b & c \\ d & e \end{pmatrix}, V^{-1} = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$, we have $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} sb & sc \\ ub & uc \end{pmatrix}$, this follows that $\begin{cases} sc = a \\ sb = 1 \end{cases}$. Since the matrix $U^{-1} = \begin{pmatrix} b & c \\ d & e \end{pmatrix}$ is invertible, hence, $bc \in V(R)$ (by Proposition 2.2). Therefore, there exists $q \in R$ such that $q + bc = 0$, this follows that $sq + sbc = sq + c = 0$ (because $sb = 1$), and so, $c \in V(R)$. By Proposition 2.1, $a = sc \in V(R)$. Thus, *R* is a ring.

$ii \Rightarrow i$: Suppose that *R* is a ring and $A=\left(a_{ij}\right)\in TM_n\left(R\right)$ is an idempotent matrix whose main diagonal has *k* entries equal to 1, and the rest equal to 0 ($0<k<n$). Then, by using elementary operations on matrix *A* as follows: If $a_{ii}=1$, multiply the entries in the i-th row by $-a_{li}$ $\left(l<i\right)$ and add to the l-th row; multiply the entries in the i-th column by $-a_{it}$ $\left(t>i\right)$ and add to the t-th column. Afterward, rearrange the rows and columns of the resulting matrix to obtain a matrix equivalent to *A* in the form $\begin{pmatrix} I_k & 0 \\ 0 & B_{n-k} \end{pmatrix}$, where all entries of the main diagonal of matrix B are equal to 0. □

Lemma 3.6. Let *R* be a commutative semiring, and $A=\left(a_{ij}\right)\in TM_n\left(R\right), n\geq 4$ be an idempotent matrix satisfying the condition $a_{ii}a_{jj}=0$ for all $i,j\in\{1,2,...,n\}$ with $i\neq j$. Then, $a_{ik}a_{kl}a_{lp}=0$ for all $i,k,l,p\in\{1,...,n\}$ with $i<k<l<p$.

Proof. If $n=4$ then $a_{12}=a_{11}a_{12}+a_{12}a_{22}=a_{12}\left(a_{11}+a_{22}\right)$, $a_{23}=a_{22}a_{23}+a_{23}a_{33}=a_{23}\left(a_{22}+a_{33}\right)$, $a_{34}=a_{33}a_{34}+a_{34}a_{44}=a_{34}\left(a_{33}+a_{44}\right)$ and $a_{12}a_{23}a_{34}=a_{12}a_{23}a_{34}\left(a_{11}+a_{22}\right)\left(a_{22}+a_{33}\right)\left(a_{33}+a_{44}\right)=0$ (because $a_{ii}a_{jj}=0,\forall i\neq j$). Assume the Lemma holds for $n=k$ $\left(k\geq 4\right)$. For any matrix $A=\left(a_{ij}\right)\in TM_{k+1}\left(R\right)$, *A* is idempotent, we rewrite the matrix *A* in the form $A=\begin{pmatrix} A_k & B \\ 0 & a_{k+1,k+1} \end{pmatrix}$ or $A=\begin{pmatrix} a_{11} & C \\ 0 & D_k \end{pmatrix}$. By Proposition 3.1, $A_k, D_k\in TM_k\left(R\right)$ are idempotent matrices. Put $r=a_{lm}a_{mq}a_{qp}$ with $1\leq l<m<q<p\leq k+1$, we have

- If $p<k+1$ then $r=a_{lm}a_{mq}a_{qp}=0$ (because a_{lm},a_{mq},a_{qp} are entries of $A_k\in TM_k\left(R\right)$).
- If $l>1$ then $r=a_{lm}a_{mq}a_{qp}=0$ (because a_{lm},a_{mq},a_{qp} are entries of $D_k\in TM_k\left(R\right)$).
- If $l=1$ and $p=k+1$ then $r=a_{1m}a_{mq}a_{q,k+1}$. We have $a_{1m}=a_{1m}\left(a_{11}+a_{mm}\right)+\sum_{h=2}^{m-1}a_{1h}a_{hm}$, $a_{q,k+1}=a_{q,k+1}\left(a_{qq}+a_{k+1,k+1}\right)+\sum_{t=q+1}^{k}a_{qt}a_{t,k+1}$. This follows that

$$r=a_{1m}a_{mq}a_{q,k+1}\left(a_{11}+a_{mm}\right)\left(a_{qq}+a_{k+1,k+1}\right)+\left(a_{11}+a_{mm}\right)\left(\sum_{t=q+1}^{k}a_{1m}a_{mq}a_{qt}a_{t,k+1}\right)$$
$$+\left(a_{qq}+a_{k+1,k+1}\right)\left(\sum_{h=2}^{m-1}a_{1h}a_{hm}a_{mq}a_{q,k+1}\right)+\sum_{h=2}^{m-1}\sum_{t=q+1}^{k}a_{1h}a_{hm}a_{mq}a_{qt}a_{t,k+1}.$$

Since $a_{ii}a_{jj}=0,\forall i\neq j$ and $a_{1m}a_{mq}a_{qt}a_{t,k+1}=0$, $a_{1h}a_{hm}a_{mq}a_{q,k+1}=0$, $a_{1h}a_{hm}a_{mq}a_{qt}a_{t,k+1}=0$ for all $h,m,q,t\in\{2,...,k\}$ with $h<m<q<t$, hence, $r=0$. □

Lemma 3.7. Let *R* be a commutative semiring, and $A=\left(a_{ij}\right)\in TM_n\left(R\right), n\geq 3$ be an idempotent matrix satisfying the condition $a_{ii}a_{jj}=0$ for all $i,j\in\{1,2,...,n\}$ with $i\neq j$. Then, $a_{lm}a_{mk}=a_{lm}a_{mm}a_{mk}$ for all $l,m,k\in\{1,...,n\}$ with $l<m<k$.

Proof. If $n=3$ then $a_{12}=a_{11}a_{12}+a_{12}a_{22}=a_{12}\left(a_{11}+a_{22}\right)$, $a_{23}=a_{22}a_{23}+a_{23}a_{33}=a_{23}\left(a_{22}+a_{33}\right)$ and $a_{12}a_{23}=a_{12}a_{23}\left(a_{11}+a_{22}\right)\left(a_{22}+a_{33}\right)=a_{12}a_{22}a_{23}$ (because $a_{ii}a_{jj}=0,\forall i\neq j$). If $n\geq 4$ then for any $l,m,k\in\{1,...,n\}$ with $l<m<k$, we have $a_{lm}=a_{lm}\left(a_{ll}+a_{mm}\right)+\sum\limits_{t=l+1}^{m-1}a_{lt}a_{tm}$ and $a_{mk}=a_{mk}\left(a_{mm}+a_{kk}\right)+\sum\limits_{s=m+1}^{k-1}a_{ms}a_{sk}$, this follows that $a_{lm}a_{mk}=a_{lm}a_{mk}\left(a_{ll}+a_{mm}\right)\left(a_{mm}+a_{kk}\right)$ $+\left(a_{ll}+a_{mm}\right)\left(\sum\limits_{s=m+1}^{k-1}a_{lm}a_{ms}a_{sk}\right)+\left(a_{mm}+a_{kk}\right)\left(\sum\limits_{t=l+1}^{m-1}a_{lt}a_{tm}a_{mk}\right)+\sum\limits_{t=l+1}^{m-1}\sum\limits_{s=m+1}^{k-1}a_{lt}a_{tm}a_{ms}a_{sk}$. By Lemma 3.6, $a_{lm}a_{mk}=a_{lm}a_{mk}\left(a_{ll}+a_{mm}\right)\left(a_{mm}+a_{kk}\right)=a_{lm}a_{mm}a_{mk}$ (because $a_{ll}a_{mm}=a_{mm}a_{kk}=0$). □

Theorem 3.8. Let *R* be a commutative semiring, and $A=\left(a_{ij}\right)\in TM_n\left(R\right)$ be an idempotent matrix satisfying the condition $a_{ii}a_{jj}=0$ for all $i,j\in\{1,2,...,n\}$ with $i\neq j$. Then, *A* is described by the following statements.

i) If $n=1$ then $A=\left(a_{11}\right)$ with $a_{11}^2=a_{11}$;

ii) If $n=2$ then $a_{ii}^2=a_{ii}, i=1,2$ and $a_{12}=a_{12}\left(a_{11}+a_{22}\right)$;

iii) If $n\geq 3$ then $a_{ii}^2=a_{ii}, i=1,...,n$ and

$$a_{ij}=\begin{cases}a_{ij}\left(a_{ii}+a_{jj}\right), j=i+1\\ a_{ij}\left(a_{ii}+a_{jj}\right)+\sum\limits_{l=i+1}^{j-1}a_{il}a_{ll}a_{lj}, j>i+1\end{cases}, \forall\ 1\leq i<j\leq n.$$

Proof. The Theorem is deduced from Remark 3.3 and Lemma 3.7. □

Consider now the structure of idempotent triangular matrices over semirings that are commutative and additively idempotent.

Proposition 3.9. Let *R* be a finite semiring that is commutative and additively idempotent with partial ordering "$\ll$", $A=\left(a_{ij}\right)\in TM_n\left(R\right)$ be an idempotent matrix with $a_{ii}\in\{0,1\}, i=1,...,n$.

i) If $n=1$ then $A=\left(a_{11}\right)$ where $a_{11}=0$ or $a_{11}=1$.

ii) If $n>1$ then entries a_{ij} $\left(1\leq i<j\leq n\right)$ are chosen as follows: If $a_{ii}+a_{jj}=0$ then $a_{ij}=\begin{cases}0, j=i+1\\ \sum\limits_{l=i+1}^{j-1}a_{il}a_{lj}, j>i+1\end{cases}$; if $a_{ii}+a_{jj}=1$ then $a_{ij}=\begin{cases}t, j=i+1\\ m, j>i+1\end{cases}$, where *t* is arbitrarily chosen from *R* and $m\in R$ such that $m\gg\sum\limits_{l=i+1}^{j-1}a_{il}a_{lj}$.

Proof. In the case of $n=1$, it is obviously. If $n>1$, then for any $i,j\in\{1,...,n\}$ with $i<j$. If $j=i+1$ then $a_{ij}=a_{ii}a_{ij}+a_{ij}a_{jj}=a_{ij}\left(a_{ii}+a_{jj}\right)=\begin{cases}0,\ a_{ii}+a_{jj}=0\\ a_{ij},\ a_{ii}+a_{jj}=1\end{cases}$. Thus, if $a_{ii}+a_{jj}=1$, the corresponding entry a_{ij} is arbitrarily chosen from *R*. If $j>i+1$ the

$$a_{ij}=a_{ij}\left(a_{ii}+a_{jj}\right)+\sum_{l=i+1}^{j-1}a_{il}a_{lj}=\begin{cases}\sum\limits_{l=i+1}^{j-1}a_{il}a_{lj},\ a_{ii}+a_{jj}=0\\ a_{ij}+\sum\limits_{l=i+1}^{j-1}a_{il}a_{lj},\ a_{ii}+a_{jj}=1\end{cases}.$$

Thus, if $a_{ii}+a_{jj}=1$, then $a_{ij}=a_{ij}+\sum\limits_{l=i+1}^{j-1}a_{il}a_{lj} \Leftrightarrow a_{ij} \gg \sum\limits_{l=i+1}^{j-1}a_{il}a_{lj}$. □

Corollary 3.10. Let *R* be a finite semiring that is commutative and additively idempotent. Then the total number of idempotent matrices $A=(a_{ij})\in TM_n(R)$ with $a_{ii}\in\{0,1\},\ i=1,...,n$ and $a_{ii}a_{jj}=0, \forall i\neq j$ is equal to $n|R|^{n-1}+1$, where $|R|$ is the number of elements in *R*.

Proof. Since $a_{ii}\in\{0,1\},\ i=1,...,n$ and $a_{ii}a_{jj}=0, \forall i\neq j$, hence, there is only one entry 1 on the main diagonal of matrix *A*. If $n=1$ or $n=2$ then it is obviously, we only need to prove the case for $n\geq 3$ as follows:

- If $a_{ii}=0,\ i=1,...,n$ then $A=(0)$ (by Proposition 3.4). (1)
- If $a_{11}=1,\ a_{ii}=0,\ i=2,...,n$, then applying Theorem 3.8 and Proposition 3.9, we conclude that a_{12} can be chosen arbitrarily from *R*, and $a_{1j}\gg\sum\limits_{l=2}^{j-1}a_{1l}a_{ll}a_{lj}=0,\ j=3,...,n$ (because $a_{11}+a_{jj}=1,\ j=2,...,n$). This follows that $a_{1j},\ j=2,...,n$ are chosen arbitrarily from *R*. Moreover, for any $i,j\in\{2,...,n\}$ with $i<j$, since $a_{ii}+a_{jj}=0$, hence, $a_{ij}=\begin{cases}0, j=i+1\\ \sum\limits_{l=i+1}^{j-1}a_{il}a_{ll}a_{lj}=0, j>i+1\end{cases}$. Thus, $A=\begin{pmatrix}1 & A'_{1\times(n-1)}\\ 0 & 0\end{pmatrix}$, where entries of $A'_{1\times(n-1)}$ are chosen arbitrarily from *R*, and so, there are $|R|^{n-1}$ ways to form matrix *A*.

(2)

- If $a_{nn}=1, a_{ii}=0,\ i=1,...,n-1$, then similarly, we obtain $A=\begin{pmatrix}0 & A''_{(n-1)\times 1}\\ 0 & 1\end{pmatrix}$, where entries of $A''_{(n-1)\times 1}$ are chosen arbitrarily from *R*, and so, there are also $|R|^{n-1}$ ways to form matrix *A*. (3)
- If $a_{kk}=1\ (1<k<n)$ and $a_{ll}=0$ for all $l\in\{1,...,n\}$ with $l\neq k$, then for any $i,j\in\{1,...,n\}$ with $i<j$, we have

- If $1\leq i<j<k$ or $k<i<j\leq n$, then from $a_{ii}+a_{jj}=0$, we also have $a_{ij}=0$.

- If $1\leq i<j=k$, then from $a_{ii}+a_{kk}=1$, we have $a_{ik}=\begin{cases}t, k=i+1\\ m, k>i+1\end{cases}$, where *t* is arbitrarily chosen from *R* and $m\in R$ such that $m\gg\sum\limits_{l=i+1}^{k-1}a_{il}a_{lk}=\sum\limits_{l=i+1}^{k-1}a_{il}a_{ll}a_{lk}=0$. This follows that entries a_{ik} are arbitrarily chosen from *R*.

- If $k=i<j\leq n$, then similarly, the entries a_{kj} are also arbitrarily chosen from *R*.

- If $1\leq i<k<j\leq n$ then $j>i+1$. Since $a_{ii}+a_{jj}=0$, hence, by Theorem 3.8 and Proposition 3.9, $a_{ij}=\sum\limits_{l=i+1}^{j-1}a_{il}a_{lj}=\sum\limits_{l=i+1}^{k-1}a_{il}a_{lj}+a_{ik}a_{kj}+\sum\limits_{l=k+1}^{j-1}a_{il}a_{lj}=a_{ik}a_{kj}$. Thus, $A=\begin{pmatrix}0 & B_{(k-1)\times 1} & BC\\ 0 & 1 & C_{1\times(n-k)}\\ 0 & 0 & 0\end{pmatrix}$, where matrices $B^T=\begin{pmatrix}a_{1k} & \cdots & a_{k-1,k}\end{pmatrix}$, $C=\begin{pmatrix}a_{k,k+1} & \cdots & a_{kn}\end{pmatrix}$ have entries chosen arbitrarily from *R*.

This implies that there are $(n-2)|R|^{n-1}$ ways to form matrix A. (4)

From (1), (2), (3), and (4), the total number of idempotent matrices $A=(a_{ij})\in TM_n(R)$ with $a_{ii}\in\{0,1\}$, $i=1,...,n$ and $a_{ii}a_{jj}=0,\forall i\neq j$ is equal to $n|R|^{n-1}+1$. □

Corollary 3.11. Let R be a finite semiring that is commutative and additively idempotent, where $|R|$ is the number of elements in R. Denote by $NTEM_n^k(R)$ the number of idempotent triangular matrices of order $n\times n$ over R, with k entries 0 and $n-k$ entries 1 on the main diagonal $(0\le k\le n)$. If $n\ge 3$ then $NTEM_n^k(R)\le \frac{n!}{k!(n-k)!}|R|^{\frac{n(n-1)}{2}-\delta_n^k},\forall 0\le k\le n-1$, where $\delta_n^k=\begin{cases}0, 0\le k<2\\ \frac{k!}{2!(k-2)!}, k\ge 2\end{cases}$. The equality sign holds if and only if $k=n-1$. In particular, if $k=n$ then $NTEM_n^n(R)=1$.

Proof. Assume that $A=(a_{ij})\in TM_n(R)$ is an idempotent matrix with k entries 0 and $n-k$ entries 1 on the main diagonal.

Case 1: If $k=0$, then $a_{ii}+a_{jj}=1,\forall 1\le i<j\le n$. By Proposition 3.9, we have $a_{ij}=\begin{cases}t, j=i+1\\ m, j>i+1\end{cases}$, where t is arbitrarily chosen from R and $m\in R$ such that $m\gg\sum_{l=i+1}^{j-1}a_{il}a_{lj}$. This implies that the number of ways to choose the entries $a_{ij}\in R$ $(1\le i<j\le n)$ does not exceed $|R|^{\frac{n(n-1)}{2}}$. Therefore, $NTEM_n^0(R)\le|R|^{\frac{n(n-1)}{2}}=\frac{n!}{0!(n-0)!}|R|^{\frac{n(n-1)}{2}-\delta_n^0}$.

Case 2: If $k=1$, then matrix A has exactly one entry 0 on the main diagonal, thus there are n ways to form the main diagonal of A. Furthermore, since $a_{ii}+a_{jj}=1,\forall 1\le i<j\le n$, hence there are no more than $|R|^{\frac{n(n-1)}{2}}$ ways to choose the entries $a_{ij}\in R$ $(1\le i<j\le n)$, and so, $NTEM_n^1(R)\le n|R|^{\frac{n(n-1)}{2}}=\frac{n!}{1!(n-1)!}|R|^{\frac{n(n-1)}{2}-\delta_n^1}$.

Case 3: If $2\le k\le n-1$, then $a_{ij}=\sum_{l=i+1}^{j-1}a_{il}a_{lj}$ (corresponding to $a_{ii}+a_{jj}=0$), so the number of such entries a_{ij} of A is $\frac{k!}{2!(k-2)!}$. This follows that there are no more than $|R|^{\frac{n(n-1)}{2}-\frac{k!}{2!(k-2)!}}=|R|^{\frac{n(n-1)}{2}-\delta_n^k}$ ways to choose the entries a_{ij} $(1\le i<j\le n)$ of A corresponding to $a_{ii}+a_{jj}=1$. Moreover, there are $\frac{n!}{k!(n-k)!}$ ways to form the main diagonal of A. Thus $NTEM_n^k(R)\le\frac{n!}{k!(n-k)!}|R|^{\frac{n(n-1)}{2}-\delta_n^k}$.

Now, if $k=n-1$, then matrix A has only one entry 1 on the main diagonal, so according to Corollary 3.10, $NTEM_n^{n-1}(R)=n.|R|^{n-1}$. On the other hand, when $k=n-1$,

$\frac{n!}{k!(n-k)!}|R|^{\frac{n(n-1)}{2}-\delta_n^k}=n.|R|^{\frac{n(n-1)}{2}-\frac{(n-1)!}{2!(n-3)!}}=n.|R|^{n-1}$. Thus the equality sign holds.

If there exists a natural number k $\left(0\le k<n-1\right)$ such that $NTEM_n^k\left(R\right)=\frac{n!}{k!(n-k)!}|R|^{\frac{n(n-1)}{2}-\delta_n^k}$. This is equivalent to every entry a_{ij} $\left(1\le i<j\le n\right)$ corresponding to $a_{ii}+a_{jj}=1$ having $|R|$ ways to be chosen from *R*. Since $0\le k<n-1$, hence the main diagonal of *A* has at least two entries 1. Suppose that $a_{ll}=a_{mm}=1$ with $1\le l<m\le n$.

- If $m=l+1$, since $n\ge 3$, without loss of generality, assume $l+1<n$. Then the entries $a_{l,l+1},a_{l+1,l+2}$ can be chosen arbitrarily from *R* (because $a_{ll}+a_{l+1,l+1}=a_{l+1,l+1}+a_{l+2,l+2}=1$). If we choose $a_{l,l+1}=a_{l+1,l+2}=1$, then the entry $a_{l,l+2}$ satisfies the condition $a_{l,l+2}\gg a_{l,l+1}a_{l+1,l+2}=1\succ 0$ (because $a_{ll}+a_{l+2,l+2}=1$), this follows that the number of ways to choose the entry $a_{l,l+2}$ is less than $|R|$ (a contradiction).
- If $m>l+1$ then entries $a_{l,l+1},a_{m-1,m}$ are chosen arbitrarily from *R* (because $a_{ll}+a_{l+1,l+1}=a_{m-1,m-1}+a_{mm}=1$). If we choose $a_{l,l+1}=a_{m-1,m}=1$, then the entry a_{lm} satisfies the condition $a_{lm}\gg\sum_{t=l+1}^{m-1}a_{lt}a_{tm}$. Furthermore, since a_{lm} is chosen arbitrarily from *R*, hence, $\sum_{t=l+1}^{m-1}a_{lt}a_{tm}=0$ implies $a_{l,l+1}a_{l+1,m}=a_{l+1,m}=0$, and so, $a_{l+1,m}$ has only one way to be chosen even though $a_{l+1,l+1}+a_{mm}=1$ (a contradiction). Thus the equality does not hold.

If $k=n$, then $a_{ii}=0,\forall i=1,...,n$. Hence, $A=\left(0\right)_n$, and thus, $NTEM_n^n\left(R\right)=1$. □

4. Conclusion

The paper has provided some properties of idempotent triangular matrices over commutative semirings, as shown in Proposition 3.1, Corollary 3.2, Proposition 3.4, and Theorem 3.5. We have described the structure of such matrices, where the entries on their main diagonal are pairwise orthogonal idempotent elements, as shown in Lemma 3.6, Lemma 3.7 and Theorem 3.8. Additionally, we have calculated the number of idempotent triangular matrices with entries on the main diagonal being 0 or 1 on finite semirings that are commutative and additively idempotent, as presented in Proposition 3.9, Corollary 3.10, and Corollary 3.11.

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