A NOTE ON THE COFINITENESS OF LOCAL COHOMOLOGY MODULES FOR A PAIR OF IDEALS

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ABSTRACT

The cofinite property plays an important role in commutative algebra. R. Hartshorne (1970) posted the question: *For which rings R and ideals I are the modules $H_I^j(N)$ is I −cofinite for all finitely generated modules N?* A similar question is raised for local cohomology modules $H_{I,J}^i(N)$ w.r.t a pair of ideals (I,J). The first aim of the note is to build a new class of modules $S_n(I,J)$ and investigate its important properties in relation to the $(I,J)-$ cofinite property of the module $H_{I,J}^i(N)$ by establishing a Grothendieck spectral sequence for the module $H_{I,J}^i(N)$. The second aim of the note is to prove the I −cofinite property of the module $H_I^i(N)$ under some conditions by applying the module class $S_n(I,0)$. With the assumption that N is an R −module such that $\mathrm{Ext}_R^j(R/I,N)$ is finitely generated for all j, this note has the first main result $H_{I,J}^{t+1}(N) \in S_n(I,J)$ if and only if $H_{I,J}^t(N) \in S_{n+2}(I,J)$, the second main result asserts that $H_I^j(N)$ is I −cofinite when I is a principal ideal and $\Gamma_I(N)$ is the module in dimension < 2. These results are more extensive than some previous results because the first result is for the extended class of local cohomology modules, and the second results for the module N which is not necessary finitely generated.

BÀI BÁO VỀ TÍNH CHẤT COFINITE CỦA MÔĐUN ĐỐI ĐỒNG ĐIỀU ĐỊA PHƯƠNG ỨNG VỚI MỘT CẶP IĐÊAN

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TỪ KHÓA

Môđun cofinite
Đối đồng điều địa phương
Đối đồng điều địa phương ứng với một cặp iđêan
Iđêan chính
Theo chiều < 2

TÓM TẮT

Tính chất cofinite có vai trò quan trọng trong đại số giao hoán. R. Hartshorne (1970) đã nêu câu hỏi: với những vành R nào và iđêan I nào làm cho môđun đối đồng điều địa phương $H_I^i(N)$ ứng với iđêan I là I −cofinite với mọi môđun hữu hạn sinh N. Một câu hỏi tương tự cũng được đặt ra cho môđun đối đồng điều địa phương $H_{I,J}^i(N)$ ứng với một cặp iđêan (I,J). Mục đích thứ nhất của bài báo là xây dựng một lớp môđun mới $S_n(I,J)$ và khảo sát tính chất quan trọng của nó trong mối liên hệ với tính chất (I,J) −cofinite của $H_{I,J}^i(N)$ bằng cách thiết lập một dãy phổ Grothendieck cho $H_{I,J}^i(N)$. Bài báo có mục đích thứ hai là xét tính chất I −cofinite của $H_I^i(N)$ trong một số điều kiện bằng cách áp dụng lớp môđun $S_n(I,0)$. Với giả thiết N là môđun làm cho $\mathrm{Ext}_R^j(R/I,N)$ hữu hạn sinh với mọi j, bài báo này có kết quả thứ nhất là $H_{I,J}^{t+1}(N) \in S_n(I,J)$ nếu và chỉ nếu $H_{I,J}^t(N) \in S_{n+2}(I,J)$, kết quả thứ hai khẳng định tính chất I −cofinite của $H_I^i(N)$ khi I là iđêan chính và $\Gamma_I(N)$ là môđun theo chiều < 2. Các kết quả này mở rộng hơn so với một số kết quả trước đây bởi vì kết quả thứ nhất dành cho lớp môđun mở rộng của môđun đối đồng điều địa phương, và kết quả thứ hai phát biểu cho môđun N không nhất thiết hữu hạn sinh.

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1. Introduction

Throughout this note the ring R is commutative Noetherian. Let j be a non-negative integer, I, J ideals of R, and N an R −module. The j^{th} local cohomology functor $H^j_{I,J}(-)$ with respect to a pair of ideals (I, J) was defined by R. Takahashi-Y. Yoshino-T. Yoshizawa [1] as the j^{th} right derived functor of (I, J) −torsion functor $\Gamma_{I,J}(-)$. They called $H^j_{I,J}(N)$ the j^{th} local cohomology module of N with respect to a pair of ideals (I, J). It is clear that $H^j_{I,0}(N)$ is just the ordinary local cohomology module $H^j_I(N)$ of N with respect to an ideal I. In [2], A. Grothendieck conjectured that $\mathrm{Hom}_R(R/I, H^j_I(N))$ is finitely generated for all j and all finitely generated module N. In [3], R. Hartshorne provided a counterexample to this conjecture; he also introduced an R −module K to be I −cofinite if $\mathrm{Supp}_R(K) \subseteq V(I)$ and $\mathrm{Ext}^j_R(R/I, K)$ is finitely generated for all j and he asked a question: *For which rings R and ideals I are the modules $H^j_I(N)$ is I −cofinite for all finitely generated modules N?* Although there have been many studies on this problem, see for example the articles [3]- [9], up to now there are still some open questions about the cofiniteness of the local cohomology module.

The first aim of this note is to investigate some questions similar to the one above for the theory of local cohomology with respect to a pair of ideals. The first main result in this note is Theorem 3.1 which shows a relation between class $S_n(I, J)$ of modules and class $S_{n+2}(I, J)$ of modules, where $S_n(I, J)$ is introduced in Definition 2.3 concerning the cofiniteness of local cohomology module $H^i_{I,J}(N)$ with respect to a pair of ideals where the module N is not necessary finitely generated. In order to prove Theorem 3.1, we need to establish a key lemma on a Grothendieck spectral sequence for local cohomology module with respect to a pair of ideals (see Lemma 2.5). In this proof we also use properties of convergent spectral sequences (refer to the book [10]) and of cofiniteness modules. In some senses, Theorem 3.1 is an extension of a result of M. Khazaei-R. Sazeedeh in [9, Proposition 2.13] for the case of local cohomology module with respect to a pair of ideals.

The second aim of this note is to apply Theorem 3.1 to investigate the cofiniteness of the ordinary local cohomology module $H^i_I(N)$ when I is a principal ideal. Hence, the second main result in this note is Theorem 3.4 which give us an affirmative answer for Hartshorne's question as mentioned above. Moreover, Theorem 3.4 is an extension of a theorem of K. I. Kawasaki [6, Theorem 1] for module of in dimension < 2 (see Corollary 3.5), where the notion of in dimension < 2 module is introduced by D. Asadollahi-R. Naghipour [11] (see Remark 3.3).

This note is divided into three sections. In Section 2, we present some necessary concepts and prove a key lemma necessary to prove the main results in this note. Section 3 devotes to prove two main theorems (Theorem 3.1 and Theorem 3.4).

2. Preliminaries

For ideals I, J of the ring R, we recall a notation $W(I, J)$ in [1, Definition 3.1] of Takahashi-Yoshino-Yoshizawa as follows

$$W(I, J) = \{p \in \mathrm{Spec}\, R \mid I^n \subseteq p + J \text{ for some } n \in \mathbb{N}\}.$$

We next recall the notion of (I, J) −cofinite module which is introduced by A. Tehranian-A. P. Eshmanan Talemi.

Definition 2.1 (see [12, Definition 2.1]). An R − module K is called $(I, J) - cofinite$ if $\mathrm{Supp}_R(K) \subseteq W(I, J)$ and $\mathrm{Ext}^j_R(R/I, K)$ is finitely generated for all j.

Remark 2.2. Since $W(I, 0) = \{p \in \mathrm{Spec}\, R \mid I^n \subseteq p + 0 \text{ for some } n \in \mathbb{N}\} = V(I) = \{p \in \mathrm{Spec}\, R \mid I \subseteq p\}$, it is clear that the notion (I, J) − cofinite module coincides to the notion I −cofinite module when $J = 0$.

We now define a class $S_n(I,J)$ of modules which is an extension of the class $S_n(I)$ of modules. The class $S_n(I)$ of modules is introduced by M. Khazaei-R. Sazeeded in [9, Definition 2.1].

Definition 2.3. Let $n \geq 0$ be an integer, and I, J ideals of R.

(i) We say that an R −module K satisfies the condition $P_n(I,J)$ if the following statement holds: *Suppose that* $\operatorname{Ext}^j_R(R/I, K)$ *is finitely generated for all* $j \leq n$ *and* $\operatorname{Supp}_R(K) \subseteq W(I,J)$. *Then the module K is* (I,J) −*cofinite*.

(ii) We define a class $S_n(I,J)$ of R −modules as follows:

$S_n(I,J) = \{K \in \mathrm{Mod} - R \mid K \text{ satisfies the condition } P_n(I,J)\}$,

where $\mathrm{Mod} - R$ is the category of modules over the ring R.

Remark 2.4. By the above definition, we observe that $S_0(I,J) \subseteq S_1(I,J) \subseteq S_2(I,J) \ldots$ Moreover, it is clear that

$$S_n(I) = \{ K \in \mathrm{Mod} - R \mid K \text{ satisfies the condition } P_n(I) \} = S_n(I,0),$$

where $P_n(I) = P_n(I,0)$.

The rest of this section, we need to establish the following key lemma on a Grothendieck spectral sequence concerning local cohomology modules with respect to a pair of ideals.

Lemma 2.5. *Let N be an* R − *module and* I, J *ideals of* R. *Then we have the following Grothendieck spectral sequence*

$$E_2^{p,q} := \operatorname{Ext}^p_R(R/I, H^q_{I,J}(N)) \Longrightarrow \operatorname{Ext}^{p+q}_R(R/I, N).$$

Proof. Note that, for an injective R −module E, we can describe the module E as a direct sum of indecomposable injective modules

$$E = \bigoplus_{p \in \operatorname{Ass}_R(E)} E(R/p)^{\mu(p,E)}$$

(cf. [13, Theorem 2.5 and Proposition 3.1]), where $E(R/p)$ is the injective hull of R −module R/p. Then, by applying (I,J) −torsion functor $\Gamma_{I,J}(-)$ for R −module E, we obtain that

$$\Gamma_{I,J}(E) = \bigoplus_{p \in \operatorname{Ass}_R(E)} \Gamma_{I,J}(E(R/p))^{\mu(p,E)}$$

Moreover, by [1, Proposition 1.11], we get that $\Gamma_{I,J}(E(R/p)) = E(R/p)$ if $p \in W(I,J)$; and $\Gamma_{I,J}(E(R/p)) = 0$ if $p \notin W(I,J)$. Hence

$$\Gamma_{I,J}(E) = \bigoplus_{p \in \operatorname{Ass}_R(E) \cap W(I,J)} \Gamma_{I,J}(E(R/p))^{\mu(p,E)}$$

Thus, we have

$$\operatorname{Ext}^j_R(R/I, \Gamma_{I,J}(E)) = \bigoplus_{p \in \operatorname{Ass}_R(E) \cap W(I,J)} \operatorname{Ext}^j_R(R/I, E(R/p))^{\mu(p,E)} = 0$$

for all $j > 0$ (here, since $E(R/p)$ is an injective R −module, the R −module $\operatorname{Ext}^j_R(R/I, E(R/p)) = 0$ for all $j > 0$). It shows that the functor $\Gamma_{I,J}(-)$ sends injective objects to Hom-acyclic objects. Moreover, note that

$$\Gamma_{I,J}(N) = \{x \in N \mid I^n \subseteq \operatorname{Ann}_R(x) + J \text{ for some } n \gg 1\},$$

we then have

$$\operatorname{Hom}_R(R/I, \Gamma_{I,J}(N)) = \operatorname{Hom}_R(R/I, N).$$

Hence, by [10, Theorem 11.38], we obtain the following Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(R/I, H^q_{I,J}(N)) \Longrightarrow \operatorname{Ext}^{p+q}_R(R/I, N),$$

as required.

3. Main results

Theorem 3.1. *Let n be non-negative integer. Let N be an $R-$module such that* $\operatorname{Ext}_R^j(R/I,N)$ *is finitely generated for all $j \geq 0$. Let t be a non-negative integer such that $H_{I,J}^i(N)=0$ for all $i \neq t, t+1$. Then $H_{I,J}^{t+1}(N) \in S_n(I,J)$ if and only if $H_{I,J}^t(N) \in S_{n+2}(I,J)$.*

Proof of Theorem 3.1. We have by Lemma 2.5 that there is a Grothendieck spectral sequence as follows

$$E_2^{p,q} \coloneqq \operatorname{Ext}_R^p(R/I, H_{I,J}^q(N)) \Longrightarrow \operatorname{Ext}_R^{p+q}(R/I,N). \quad (1)$$

For each integer p, we have the chain complex

$$E_2^{p-2,t+1} \xrightarrow{d_2^{p-2,t+1}} E_2^{p,t} \xrightarrow{d_2^{p,t}} E_2^{p+2,t-1} \quad (2)$$

By the hypothesis of $H_I^i(N)$ we get $E_2^{p+2,t-1} = \operatorname{Ext}_R^{p+2}\left(R/I, H_{I,J}^{t-1}(N)\right) = 0$. It implies that

$$E_3^{p,t} = \frac{\operatorname{Ker}\left(d_2^{p,t}\right)}{\operatorname{Im}\left(d_2^{p-2,t+1}\right)} = E_2^{p,t}/\operatorname{Im}(d_2^{p-2,t+1}). \quad (3)$$

Consider the chain complex

$$E_3^{p-3,t+2} \xrightarrow{d_3^{p-3,t+2}} E_3^{p,t} \xrightarrow{d_3^{p,t}} E_3^{p+3,t-2}.$$

Note that by the hypothesis we have

$$E_2^{p-3,t+2} = \operatorname{Ext}_R^{p-3}(R/I, H_{I,J}^{t+2}(N)) = 0 \text{ and } E_2^{p+3,t-2} = \operatorname{Ext}_R^{p+3}(R/I, H_{I,J}^{t-2}(N)) = 0.$$

Hence, since $E_3^{p-3,t+2}$ and $E_3^{p+3,t-2}$ are subquotients of $E_2^{p-3,t+2}$ and $E_2^{p+3,t-2}$, respectively, we obtain that $E_3^{p-3,t+2}=0$ and $E_3^{p+3,t-2}=0$. Therefore

$$E_4^{p,t} = \frac{\operatorname{Ker}\left(d_3^{p,t}\right)}{\operatorname{Im}\left(d_3^{p-3,t+2}\right)} = \frac{E_3^{p,t}}{0} = E_3^{p,t}.$$

By continuing this process, we have that $E_\infty^{p,t} = E_4^{p,t} = E_3^{p,t}$. We then have by (2) and (3) the following exact sequence

$$E_2^{p-2,t+1} \xrightarrow{d_2^{p-2,t+1}} E_2^{p,t} \to E_\infty^{p,t} (= E_3^{p,t} = E_2^{p,t}/\operatorname{Im}(d_2^{p-2,t+1})) \to 0. \quad (4)$$

We next consider the chain complex

$$E_2^{p-2,t+2} \xrightarrow{d_2^{p-2,t+2}} E_2^{p,t+1} \xrightarrow{d_2^{p,t+1}} E_2^{p+2,t}.$$

By the hypothesis we have $E_2^{p-2,t+2} = \operatorname{Ext}_R^{p-2}\left(R/I, H_{I,J}^{t+2}(N)\right) = 0$. Hence

$$E_3^{p,t+1} = \operatorname{Ker}\left(d_2^{p,t+1}\right) \subseteq E_2^{p,t+1}.$$

Consider the following chain complex

$$E_3^{p-3,t+3} \xrightarrow{d_3^{p-3,t+3}} E_3^{p,t+1} \xrightarrow{d_3^{p,t+1}} E_3^{p+3,t-1},$$

where $E_3^{p+3,t-1} = E_3^{p-3,t+3} = 0$ by the hypothesis of $H_{I,J}^j(N)$ with $j=t-1$ or $j = t+3$. It yields that $E_4^{p,t+1} = E_3^{p,t+1}$. By continuing in the same way, we then get that $E_\infty^{p,t+1} = E_4^{p,t+1} = E_3^{p,t+1}$. Thus, we obtain the following exact sequence

$$0 \to E_\infty^{p,t+1} (= \operatorname{Ker}(d_2^{p,t+1})) \to E_2^{p,t+1} \xrightarrow{d_2^{p,t+1}} E_2^{p+2,t}. \quad (5)$$

For any non-negative integer v, by the convergent spectral sequence (1), there is a finite filtration of the module $H^v \coloneqq \operatorname{Ext}_R^v(R/I,N)$ as follows

$$0 = \phi^{v+1}H^v \subseteq \phi^v H^v \subseteq \ldots \subseteq \phi^1 H^v \subseteq \phi^0 H^v = H^v$$

in which $E_\infty^{i,v-i} \cong \phi^i H^v/\phi^{i+1}H^v$ for all $0 \leq i \leq v$. In particular, we obtain that the modules $E_\infty^{p,t}$ and $E_\infty^{p,t+1}$ are subquotients of modules $\operatorname{Ext}_R^{p+t}(R/I,N)$ and $\operatorname{Ext}_R^{p+t+1}(R/I,N)$, respectively. Thus $E_{\infty}^{p,t}$ and $E_{\infty}^{p,t+1}$ are finitely generated for all p (since $\operatorname{Ext}_R^j(R/I,N)$ is finitely generated for all j by the hypothesis).

We now assume that $H_{I,J}^t(N) \in S_{n+2}(I,J)$ and $\operatorname{Ext}_R^p(R/I, H_{I,J}^{t+1}(N))$ is finitely generated for all $p \leq n$. We need to claim that $H_{I,J}^{t+1}(N)$ is $(I,J)-$cofinite (and so $H_{I,J}^{t+1}(N) \in S_n(I,J)$). Indeed, we have by the assumption that $E_2^{p,t+1}$ is finitely generated for all $p \leq n$. Thus $E_2^{p-2,t+1}$ is finitely generated for all $p \leq n+2$. Hence, we get by the exact sequence (4) that $E_2^{p,t}$ is finitely generated for all $p \leq n+2$ (since $E_{\infty}^{p,t}$ is finitely generated for all p as shown in previous paragraph). In other words, $\operatorname{Ext}_R^p(R/I, H_{I,J}^t(N)) = E_2^{p,t}$ is finitely generated for all $p \leq n+2$. This follows that $H_{I,J}^t(N)$ is $(I,J)-$cofinite by the assumption $H_{I,J}^t(N) \in S_{n+2}(I,J)$. Hence $E_2^{p+2,t} = \operatorname{Ext}_R^{p+2}(R/I, H_{I,J}^t(N))$ is finitely generated for all p. Thus, we have by the exact sequence (5) that $E_2^{p,t+1} = \operatorname{Ext}_R^p(R/I, H_{I,J}^{t+1}(N))$ is finitely generated for all p (keep in mind that, in (5), the module $E_{\infty}^{p,t+1}$ is finitely generated for all p). This ensures that $H_{I,J}^{t+1}(N)$ is $(I,J)-$cofinite. Hence, the claim is proved.

Conversely, assume that $H_{I,J}^{t+1}(N) \in S_n(I,J)$ and $\operatorname{Ext}_R^p(R/I, H_{I,J}^t(N))$ is finitely generated for all $p \leq n+2$. We need to show that $H_{I,J}^t(N)$ is $(I,J)-$cofinite (and so $H_{I,J}^t(N) \in S_{n+2}(I,J)$). We have by the assumption that $E_2^{p,t} = \operatorname{Ext}_R^p(R/I, H_{I,J}^t(N))$ is finitely generated for all $p \leq n+2$. Thus $E_2^{p+2,t}$ is finitely generated for all $p \leq n$. From this and the exact sequence (5), we obtain that $E_2^{p,t+1} = \operatorname{Ext}_R^p(R/I, H_{I,J}^{t+1}(N))$ is finitely generated for all $p \leq n$. Hence $H_{I,J}^{t+1}(N)$ is $(I,J)-$cofinite by the assumption that $H_{I,J}^{t+1}(N) \in S_n(I,J)$. It yields that $\operatorname{Ext}_R^j(R/I, H_{I,J}^{t+1}(N))$ is finitely generated for all j. Hence $E_2^{p-2,t+1} = \operatorname{Ext}_R^{p-2}(R/I, H_{I,J}^{t+1}(N))$ is finitely generated for all p. It follows by the exact sequence (4) that $E_2^{p,t} = \operatorname{Ext}_R^p(R/I, H_{I,J}^t(N))$ is finitely generated for all p. That is, the module $H_{I,J}^t(N)$ is $(I,J)-$cofinite. Hence $H_{I,J}^t(N) \in S_{n+2}(I,J)$, and the proof of our theorem is completed.

Recall that $W(I,0) = V(I) = \{p \in \operatorname{Spec} R \mid I \subseteq p\}$, $P_n(I,0) = P_n(I)$, $S_n(I,0) = S_n(I) = \{K \in \operatorname{Mod} - R \mid K \text{ satisfies the condition } P_n(I)\}$ and $H_{I,0}^i(-) = H_I^i(-)$ for all i. Therefore, by replacing $J = 0$ in Theorem 3.1, we obtain the following corollary on the ordinary local cohomology modules with respect to an ideal.

Corollary 3.2. *Let* n *be an integer with* $n \geq 0$. *Suppose that* $\operatorname{Ext}_R^j(R/I,N)$ *finitely generated for all* $j \geq 0$. *Assume that* $H_I^i(N) = 0$ *for all* $i \neq t, t+1$ *for some given non-negative integer* t. *Then* $H_I^{t+1}(N) \in S_n(I)$ *if and only if* $H_I^t(N) \in S_{n+2}(I)$.

Note that there is a slight difference in the statement of [9, Proposition 2.13] and Corollary 3.2. In our opinion, the arguments of Khazaei-Sazeedeh in the proof of [9, Proposition 2.13] are correct for the hypothesis that $\operatorname{Ext}_R^j(R/I,N)$ is finitely generated for all $j \geq 0$ but not for the hypothesis as stated in [9]. So Corollary 3.2 here can be regarded as a slight correction for the statement of [9, Proposition 2.13].

Remark 3.3. Before consider the next consequence of Theorem 3.1 and Corollary 3.2, we need to recall that an $R-$module K is called in dimension < 2 if there exists a finitely generated submodule T of K such that $\operatorname{dimSupp}_R(K/T) < 2$ (that is, $\dim(R/p) \leq 1$ for all $p \in \operatorname{Supp}_R(K/T)$) (see [11, Definition 2.1] of Asadollahi-Naghipour). It is clear that the class of in dimension < 2 modules consist of the class of finitely generated modules and the class of $R-$modules K with $\operatorname{dimSupp}_R(K) \leq 1$.

The following theorem on the cofiniteness of local cohomology module $H_I^i(N)$ with respect to an ideal which gives us an affirmative answer for a question of Hartshorne as shown in Introduction part.

Theorem 3.4. *Let I be a principal ideal of R and N an R −module such that* $\mathrm{Ext}_R^j(R/I,N)$ *is finitely generated for all j. Assume that the module $H_I^0(N)$ is in dimension <2. Then $H_I^i(N)$ is I −cofinite for all $i \geq 0$.*

Proof of Theorem 3.4. Since I is a principal ideal, $H_I^i(N) = 0$ for all $i \neq 0,1$. Thus, we obtain by Corollary 3.2 that $H_I^0(N) \in S_2(I)$ if and only if $H_I^1(N) \in S_0(I)$ (*). Since the modules $\mathrm{Ext}_R^j(R/I,N)$ is finitely generated for all j, and the module $H_I^0(N) = \Gamma_I(N)$ is in dimension < 2 by the hypothesis, we obtain by [8, Theorem 1.1] that the module $H_I^0(N)$ is I −cofinite. Thus, $\mathrm{Ext}_R^j(R/I,H_I^0(N))$ is finitely generated for all $j \geq 0$, and hence the module $H_I^0(N) \in S_2(I)$. It implies that $H_I^1(N)$ belongs to class $S_0(I)$ by (*). Thus, in order to prove the cofiniteness of local cohomology module $H_I^1(N)$ with respect to an ideal I, we need only to show that the module $\mathrm{Ext}_R^0(R/I,H_I^1(N))$ is finitely generated. Finally, to do this, we apply [8, Theorem 1.1] once again, we then obtain that the R −module $\mathrm{Hom}_R(R/I,H_I^1(N))$ is finitely generated. Therefore the R −module $H_I^1(N)$ is I −cofinite, as required.

Note that if an R −module M is finitely generated then M also is in dimension <2 module. Therefore, as an immediately consequence of Theorem 3.4, we obtain the following result of K. I. Kawasaki in [6].

Corollary 3.5 (see [6, Theorem 1]). *Let I be a principal ideal and M a finitely generated R −module. Then the local cohomology module $H_I^i(M)$ is I −cofinite for all $i \geq 0$.*

4. Conclusion

We have proven the first new result on the relation between class $S_n(I,J)$ of modules and class $S_{n+2}(I,J)$ of modules concerning the cofiniteness of local cohomology module $H_{I,J}^i(N)$ with respect to a pair of ideals (I,J) in this note by using properties of a Grothendieck spectral sequence and of cofiniteness modules. Furthermore, as an application of the above result and our recent result [8], we obtain the second new result on the cofiniteness of local cohomology modules with respect to a principal ideal which covers a theorem of K. I. Kawasaki in [6]. In future, we will consider more applications of Theorem 3.1 and Theorem 3.4 to study the cofiniteness of local cohomology modules with respect to a pair of ideals in some certain conditions.

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