

STABILIZING CONTROL FOR AN ECCENTRIC PENDULUM SYSTEM BASED ON ENERGY METHOD

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ARTICLE INFO	ABSTRACT
Received: 10/11/2025 Revised: 20/4/2026 Published: 20/4/2026	This paper proposes an adaptive control method to stabilize a cart-inverted pendulum system with an eccentric mass. The displacement of the center of mass along the pendulum increases the system's nonlinearity and dynamic complexity, posing challenges for both stabilization and motion control. The nonlinear dynamic model is formulated using the Euler–Lagrange approach, accurately representing the effects of eccentric mass and external disturbances. An energy-based adaptive controller is then designed to enable online parameter estimation while ensuring system stability and robustness. Simulation results demonstrate that the system achieves fast and stable convergence, exhibits strong disturbance rejection capability, and maintains control inputs within acceptable limits. The proposed control scheme can be effectively applied to swing-up control of pendulums, stabilization of self-balancing two-wheeled robots, and other nonlinear electromechanical systems. This study contributes a practical and efficient framework for controlling underactuated nonlinear systems with uncertain parameters.
KEYWORDS	
Cart-inverted pendulum system	
Eccentric pendulum	
Energy control	
Online parameter estimation	
Underactuated system	

ĐIỀU KHIỂN ỔN ĐỊNH HỆ CON LẮC LỆCH TÂM SỬ DỤNG PHƯƠNG PHÁP NĂNG LƯỢNG

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THÔNG TIN BÀI BÁO	TÓM TẮT
Ngày nhận bài: 10/11/2025 Ngày hoàn thiện: 20/4/2026 Ngày đăng: 20/4/2026	Bài báo này đề xuất một phương pháp điều khiển thích nghi nhằm ổn định hệ xe – con lắc ngược có khối lượng lệch tâm. Sự dịch chuyển của tâm khối lượng trên thanh lắc làm tăng tính phi tuyến và độ phức tạp động học của hệ, gây khó khăn cho điều khiển ổn định và chuyển động của hệ. Mô hình phi tuyến của hệ được xây dựng theo phương pháp Euler–Lagrange, phản ánh chính xác ảnh hưởng của khối lượng lệch tâm và nhiễu bên ngoài. Bộ điều khiển thích nghi theo năng lượng được thiết kế, cho phép ước lượng tham số trực tuyến và đảm bảo tính ổn định, bền vững của hệ. Kết quả mô phỏng cho thấy hệ đạt được trạng thái ổn định nhanh, khả năng chống nhiễu tốt và tín hiệu điều khiển luôn nằm trong giới hạn cho phép. Phương pháp đề xuất có khả năng ứng dụng trong điều khiển nâng con lắc (swing-up), ổn định robot hai bánh tự cân bằng và các hệ cơ điện phi tuyến khác. Nghiên cứu này góp phần xây dựng một hướng tiếp cận hiệu quả cho điều khiển các hệ phi tuyến thiếu cơ cấu chấp hành có tham số bất định.
TỪ KHÓA	
Hệ xe - con lắc ngược	
Hệ con lắc ngược lệch tâm	
Điều khiển dựa trên năng lượng	
Ước lượng tham số trực tuyến	
Hệ thiếu cơ cấu chấp hành	

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1. Introduction

The cart and pole model is considered as the simplest form of the inverted pendulum system, utilizing horizontal motion to balance the pendulum at the inherently unstable upright position. The earliest inverted pendulum system, known as Kapitza's model [1], achieves stabilization at the inverted equilibrium by applying high-frequency vertical oscillations to the pivot point. Other notable inverted pendulum configurations include the rotary inverted pendulum [2], which uses rotational motion for stabilization, and two wheeled mobile robots [3] such as Dean Kamen's iBot [4], which employ wheel movement to navigate terrain, enabling mobility for patients, individuals, or even robotic platforms in games. Additionally, models like the BallBot [5], which balances on a single spherical wheel, or models utilizing mechanical limbs, often seen in walking robots such as BigDog [6] and humanoid robots [7], are all representations of the inverted pendulum principle in reality.

The cart and pole model is an under-actuated system that has been chosen by researchers of control theory and kinematics as the research object to analyze kinematic behavior, dynamic response, and test control algorithms. In the case of an eccentric cart and pole system, the tilt angle $\theta = \pi$ (Figure 1) may not be a stable equilibrium due to asymmetrical mass distribution. The control design approaches the system's kinematics and states using both linear and nonlinear techniques such as PID, LQR [8], adaptive control [9], intelligent control [10], robust control [11],..., or energy-based control [12] methods. In the energy-based approach, the total mechanical energy at equilibrium is given by: $E^d = m_N g l_N$ where m_N and l_N are the total mass and the distance from the pivot to the pendulum's center of mass, respectively. These values are often difficult to determine in practice, as they are unknown and random. By setting the initial energy reference as $E^d = m_N g l_N$, assuming no eccentric mass, the system can still ensure stability within a certain range.

In this paper, a control strategy for stabilizing the eccentric inverted pendulum system is designed and simulated to evaluate the robustness and performance of the proposed controller.

2. Materials and Methods

2.1. System Modeling

A cart and pole model with eccentric mass and its parameters are displayed in Figure 1 and Table 1. From Figure 1, it can be concluded that with an added eccentric mass the new center of mass is changed and its new direction is not consistent with the pendulum.

Table 1. The cart and pole variables and parameters

Parameter	Description
F	The external force.
m_c, m_1, m_i	The mass of the cart, pole, eccentric mass.
θ	The counter-clockwise angle of the pole; zero is hanging straight down.
$\Delta\theta$	The angle between the new center of mass and the pole.
$\omega, \dot{\theta}$	The angular velocity of the pole.
$\alpha, \ddot{\theta}$	The angular acceleration of the pole.
x	The cart displacement along the x-axis.
g	The gravitational acceleration.
l_1, l_i	The center of mass of the pole and eccentric mass distance.
I_1, I_i	The pole and eccentric mass inertia moment.
m_n, l_n, I_n	The mass, distance and inertia moment of the added eccentric mass.

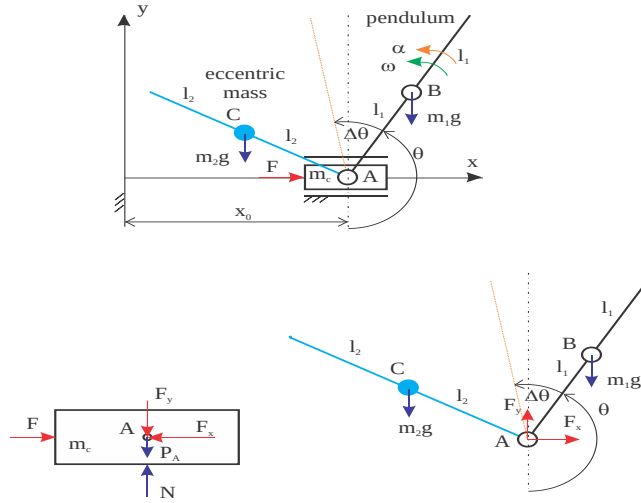


Figure 1. The cart and pole system and its parameters

The equations of the system are illustrated as:

$$(m_c + m_N)\ddot{x} + m_N l_N \ddot{\theta} \cos(\theta + \Delta\theta) - m_N l_N \dot{\theta}^2 \sin(\theta + \Delta\theta) = F$$

$$m_N l_N \ddot{x} \cos(\theta + \Delta\theta) + (I_N + m_N l_N^2) \ddot{\theta} = m_N l_N g \sin(\theta + \Delta\theta) \quad (1)$$

Where the new eccentric pole and its parameters are calculated as:

$$m_N = \sum m_i$$

$$l_N = \frac{1}{m_N} \sum m_i l_i \quad (2)$$

$$I_N = \sum I_i$$

The energy is given by

$$T = \frac{1}{2} (m_c + m_N) \dot{x}^2 + m_N \dot{x} \dot{\theta} l_N \cos(\theta + \Delta\theta) + \frac{1}{2} m_N l_N^2 \dot{\theta}^2$$

$$U = -m_N g l_N \cos(\theta + \Delta\theta) \quad (3)$$

From (1), the tilt angle is expressed as

$$\ddot{\theta} = a F_x \cos(\theta + \Delta\theta) + a g \sin(\theta + \Delta\theta) \quad (4)$$

$$\text{Where } a = \frac{m_N l_N}{I_N + m_N l_N^2}$$

2.2. Control design

Assuming that all the states of the system are bounded and measurable. At the initial time t_0 , the pole lies strictly in the upper half plane and there exists a positive constant c_0 such that $|\theta(t_0)| \leq c_0$. Then, the stability problem is to find the control force F to bring the pole to equilibrium and keep it stable in that position while tracking the reference trajectories or serial points.

With the cart and pole system, if all the states and parameters can be measured and known, to stabilize the pole at the equilibrium, inverse dynamic or energy methods can be used in combination with control design techniques to accomplish this task. The problem arises when an eccentric mass is added to the pole, in which case the parameters of the system become unknown; the equilibrium position ($\theta=\pi$) is transformed into an unknown value ($\theta + \Delta\theta$), making the control design process for this system more difficult. Using direct parameter estimation and adaptive control laws is one of the control design methods applied to this problem.

By defining $\hat{\theta}$ as the tilt angle created by the line passing through the new center of gravity of the pole and the vertical axis, the task is to stabilize the eccentric pole at the unstable fixed point $\hat{\theta} = \pi, x = 0; \dot{\theta} = \dot{\hat{\theta}} = 0; \dot{x} = 0$.

$$\text{The total energy of the pole is given by: } U = \frac{1}{2} (I_N + m_N l_N^2) \dot{\theta}^2 + m_N g l_N (1 - \cos \hat{\theta}) \quad (5)$$

$$\text{The desired energy at the fixed point is: } U_d = m_N g l_N (1 - \cos \pi) = 2 m_N g l_N \quad (6)$$

Defining the energy error $U_e = U - U_d$ and differentiating it with respect to time and (1) as:

$$\dot{U}_e = -m_N l_N F_x \dot{\theta} \cos \hat{\theta} \quad (7)$$

Where $F_x = m_1 \ddot{x}$.

Choosing $F_x = k_1 \dot{\theta} \cos \hat{\theta} U_e$ for $k_1 > 0$

Then the differential energy error is: $\dot{U}_e = -k_1 m_N l_N \dot{\theta}^2 \cos^2 \hat{\theta} U_e < 0$.

This error implies an exponential convergence: $U_e \rightarrow 0$.

Choosing $\sigma(x) = x(3d^2 - x^2)/(2d^3)$ as a smooth saturation and p-times function in which d is the maximum value dependent on the limited forces generated by the actuator of the cart and pole system. F_x can be rewritten as: $F_x = \sigma(k_1 \dot{\theta} \cos \hat{\theta} U_e)$ (8)

Defining errors for tracking trajectories or serial points as $x_{1e} = x - x_d$ and $x_{2e} = \dot{x} - \dot{x}_d$, where x and x_d are the displacement states and the references.

The virtual control and control force are chosen as: $\alpha_2 = -k_2 x_{1e}$ (9)
 $F = -k_3 x_{2e} - x_{1e} - F_x$

$$(m_c + m_N) \ddot{x} = F + h(\hat{\theta}_1, \hat{\theta}_2, \hat{a})$$

Where $\hat{\theta}_1 = \dot{\theta}, \hat{\theta}_2 = \ddot{\theta}$ (10)
 $\hat{a} = -\dot{\theta}^2 - k_4 \tilde{a}$

3. Results and Discussion

The parameters for simulation are taken from [13] as follows: $m_1 = 1[kg]$, $m_2 = 1[kg]$, $m_c = 3,5[kg]$, $g = 9,8[m/s^2]$, $l_1 = 0,5[m]$, and $I_1 = 0,33[kg.m^2]$; $l_2 = 0,5[m]$ and $I_2 = 0,33[kg.m^2]$. The initial conditions are taken as $\theta(0) = 1[rad]$, $\dot{\theta}(0) = 0,2[rad/s]$, $x(0) = -1[m]$, and $\dot{x}(0) = 0[m/s]$. The control gains $k_1 = 2, k_2 = 5, k_3 = 3, k_4 = 9$ are chosen.

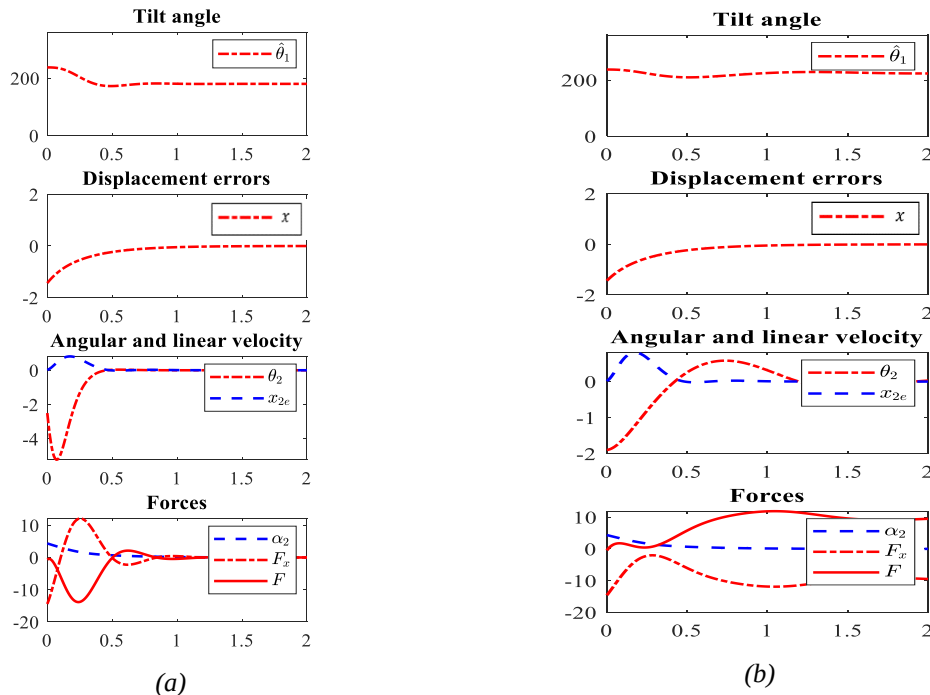


Figure 2. The simulation for the cart and pole system (a) without and (b) with the eccentric mass

The simulation results for the designed control algorithms in two scenarios, with and without an eccentric mass added to the pole, are illustrated in Figure 2. In the first case, when no eccentric mass is present, the pole needs to stabilize at the vertical position where the tilt angle θ is 180° . As shown in Figure 2a, the system stabilizes the pole at the upright equilibrium position

corresponding to the angle $\hat{\theta}_1 = \pi = 180^\circ$. In the second case, when an eccentric mass ($m_2 = 1$ [kg], $l_2 = 0,5$ [m]) is attached to the pendulum as shown in Figure 1, a new center of mass of the eccentric pendulum is created. It can be calculated that the new tilt angle to stabilize the pole at $\hat{\theta}_1 = \theta + \Delta\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4} = 225^\circ$. It can be seen from Figure 2b that the system reaches a new equilibrium at $\hat{\theta}_1 = 5\pi/4$ after about 0.6 seconds. These results are consistent with theoretical predictions: in the absence of an eccentric mass, the system stabilizes vertically with the pole aligned along the direction passing through its center of mass. As shown in Figure 2b, when the eccentric mass is added, the new center of mass shifts, resulting in the new direction deviating by an angle of $\Delta\theta = \pi/4$. The simulation shows that the tracking error in displacement converges toward zero as $x \rightarrow x_d$, and the control forces remain within the predefined actuator limits and do not exceed the maximum allowable values of the actuator.

4. Conclusion

In this paper, a method for designing a controller to stabilize the eccentric cart and pole system is illustrated. The new desired energy, when adding the eccentric mass, is calculated via equation (6), (2) and is estimated online via equation (10) for the control design based on the energy approach. The simulation shows that, the system is stable with or without an eccentric mass. These results can be extended and applied to swing the eccentric pendulum up in the future work.

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