

## HYBRID PROJECTION ALGORITHM FOR SPLIT VARIATIONAL INEQUALITIES WITH MULTIPLE OUTPUT SETS IN HILBERT SPACES

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| ARTICLE INFO                 |           | ABSTRACT                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
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| <b>Received:</b>             | 27/3/2026 | In this paper, we study the split variational inequality problem with multiple output sets in real Hilbert spaces. This problem provides a unified framework that includes several important split models as special cases, such as the split feasibility problem, the split common minimum point problem, and the split common fixed-point problem with multiple output sets. To solve this problem, we propose a novel iterative algorithm that combines the hybrid projection technique with a multistep inertial extrapolation. The algorithm is designed with self-adaptive step sizes and control parameters, and therefore does not require any prior knowledge of the norms of the transfer operators. Under suitable assumptions, we prove that the sequence generated by the proposed algorithm converges strongly to a solution of the problem. Our results also extend and improve several existing works in the literature by removing the restrictive requirement to compute or estimate the norms of the transfer operators in previous algorithms. |
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| <b>KEYWORDS</b>              |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| Hilbert space                |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| Monotone operator            |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| Split feasibility problem    |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| Split variational inequality |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| Hybrid projection algorithm  |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |

## THUẬT TOÁN CHIỀU LẠI GHEP CHO BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN TÁCH VỚI ĐA TẬP ĐẦU RA TRONG KHÔNG GIAN HILBERT

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| THÔNG TIN BÀI BÁO            |           | TÓM TẮT                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
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| <b>Ngày nhận bài:</b>        | 27/3/2026 | Trong bài báo này, chúng tôi nghiên cứu bài toán bất đẳng thức biến phân tách với đa tập đầu ra trong không gian Hilbert thực. Bài toán này cung cấp một khung thống nhất bao gồm một số mô hình tách quan trọng như các trường hợp đặc biệt, chẳng hạn như bài toán chấp nhận tách, bài toán điểm tiêu chung tách và bài toán điểm bất động chung phân tách với đa tập đầu ra. Để giải bài toán này, chúng tôi đề xuất một thuật toán lặp mới kết hợp kỹ thuật chiếu lại ghép với phép ngoại suy quán tính đa bước. Thuật toán được thiết kế với kích thước bước và tham số điều khiển có thể tự thích ứng, do đó không yêu cầu bất kỳ kiến thức nào về chuẩn của các toán tử chuyển. Dưới các giả thiết phù hợp, chúng tôi chứng minh rằng dãy sinh ra từ thuật toán đề xuất hội tụ mạnh đến nghiệm của bài toán. Kết quả của chúng tôi cũng mở rộng và cải thiện một số công trình hiện có trong tài liệu bằng cách loại bỏ yêu cầu hạn chế phải tính toán hoặc ước lượng chuẩn của các toán tử chuyển trong các thuật toán trước đó. |
| <b>Ngày hoàn thiện:</b>      | 20/5/2026 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| <b>Ngày đăng:</b>            | 20/5/2026 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| <b>TỪ KHÓA</b>               |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Không gian Hilbert           |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Toán tử đơn điệu             |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Bài toán chấp nhận tách      |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Bất đẳng thức biến phân tách |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Thuật toán chiếu lại ghép    |           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |

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## 1. Introduction

The split feasibility problem (SFP) was first introduced by Censor et al. [1], and has since become a fundamental topic of interest in applied mathematics. Due to its broad applicability in fields such as signal processing, image reconstruction [2], and intensity-modulated radiation therapy [3], it has received significant attention from the research community. Over time, this problem has been extensively studied and generalized in various directions within more abstract frameworks.

In many practical applications, particularly in signal recovery and image reconstruction, one often has to deal with multiple sensors, observation channels, or measurement devices rather than a single output system. This practical requirement has led to the development of split models involving multiple operators. This line of research was initiated by Reich et al. [4], who studied a generalized split model known as the split feasibility problem with multiple output sets (SFP-MOS). Since then, this framework has been generalized in various directions, and accordingly, several iterative algorithms have also been proposed for the corresponding models. For more details, we refer the reader to [4] – [13] and references therein.

Within this general framework, in the present paper we consider the split variational inequality problem with multiple output sets (SVI-MOS), which is formulated as follows. Assume that:

- (C1)  $H$  and  $H_i$  are real Hilbert spaces for each  $i \in \mathcal{J}_N := \{1, 2, \dots, N\}$ ;  $C$  and  $C_i$  are nonempty, closed, and convex subsets of  $H$  and  $H_i$ , respectively.
- (C2)  $A : H \rightarrow H$  is an  $\alpha$ -inverse strongly monotone operator and, for each  $i \in \mathcal{J}_N$ ,  $A_i : H_i \rightarrow H_i$  is an  $\alpha_i$ -inverse strongly monotone operator;
- (C3)  $T_i : H \rightarrow H_i$  ( $i \in \mathcal{J}_N$ ) are bounded linear operators;
- (C4)  $\Delta := \Delta_A \cap (\cap_{i=1}^N T_i^{-1}(\Delta_{A_i})) \neq \emptyset$ , where

$$\Delta_A = \{x \in C : \langle A(x), z - x \rangle \geq 0, \forall z \in C\}$$

and

$$\Delta_{A_i} = \{\bar{x} \in C_i : \langle A_i(\bar{x}), z - \bar{x} \rangle \geq 0, \forall z \in C_i\}.$$

Given these data, the SVI-MOS consists of finding an element  $x_* \in \Delta$ .

It is worth noting that the SVI-MOS framework covers several important classes of split problems as special cases, including the SFP-MOS, the split common minimum point problem with multiple output sets (SCMPP-MOS), and the split common fixed point problem with multiple output sets (SCFPP-MOS) (see Remark 3.2-Remark 3.4). Consequently, SVI-MOS can be regarded as a unified and general split problem model, and algorithms developed for solving it can be applied to a wide range of related split problems.

Motivated by the above discussion and building on hybrid projection techniques, in this paper, we propose a new algorithm for solving the SVI-MOS. The proposed method incorporates a multistep inertial extrapolation strategy to enhance the convergence rate, while the associated control parameters are selected in a self-adaptive manner (see Section 3). The main contributions of this paper can be summarized as follows.

- (1) We introduce a new algorithm, called ALGO-IMHPA (see Section 3), and establish that the iterative sequences generated by the algorithm converge strongly to a solution of the SVI-MOS.
- (2) The step-size and control parameters are determined in a self-adaptive manner and without requiring prior knowledge of the norms of the transfer operators.
- (3) As applications, we derive corresponding results for several important special cases, including the SFP-MOS, the SCMPP-MOS, and the SCFPP-MOS; see Remarks 3.2–3.4. In addition, these results overcome certain limitations of earlier approaches, in which one must compute, or at least estimate, the norms of the operators  $T_i$ , as required in [4, Algorithms (1.5), (1.6)], [5, Algorithm 1], [10, Algorithm (9)], and [14, Algorithms (3.1), (3.5)].

## 2. Preliminaries

Let  $C$  be a nonempty, closed, and convex subset of a real Hilbert space  $H$ . We denote by “ $\rightarrow$ ”, “ $\rightharpoonup$ ”, “ $\langle \cdot, \cdot \rangle$ ”, and “ $\| \cdot \|$ ” the strong convergence, weak convergence, the inner product, and the norm on  $H$ , respectively.

An operator  $A : H \rightarrow H$  is called nonexpansive if  $\|A(x) - A(y)\| \leq \|x - y\|$ ,  $\forall x, y \in H$ ; monotone if  $\langle A(x) - A(y), x - y \rangle \geq 0$ ,  $\forall x, y \in H$ ; and  $\alpha$ -inverse strongly monotone if there exists  $\alpha > 0$  such that  $\langle A(x) - A(y), x - y \rangle \geq \alpha \|A(x) - A(y)\|^2$ ,  $\forall x, y \in H$ .

The mapping  $P_C : H \rightarrow C$ , defined by  $P_C(x) = y$  if  $y$  satisfies  $\|y - x\| \leq \|z - x\|$ ,  $\forall z \in C$ , is called the metric projection of  $H$  onto  $C$ . It is well known that  $P_C$  is nonexpansive on  $H$  and that  $C = \text{Fix}(P_C) := \{x \in H : x = P_C(x)\}$ .

The following lemmas will be useful in the proofs of our main results.

**Lemma 2.1.** (see [15, Lemma 2.10])

If  $A : H \rightarrow H$  is an  $\alpha$ -inverse strongly monotone operator, then the mapping  $P_C(I - \rho A)$  is nonexpansive for every  $\rho \in (0, 2\alpha)$ , where  $I$  is the identity mapping on  $H$ .

**Lemma 2.2.** (see [16, Theorem 3.14] and [9, Lemma 2.5])

Let  $P_C$  be the metric projection from  $H$  onto  $C$ . Then the following statements hold:

- (1)  $\langle u - P_C(u), z - P_C(u) \rangle \leq 0$  for all  $u \in H$  and  $z \in C$ .
- (2)  $\|u - P_C(u)\|^2 + \|z - P_C(u)\|^2 \leq \|z - u\|^2$  for all  $u \in H$  and  $z \in C$ .

**Lemma 2.3.** Let  $A : H \rightarrow H$  be an operator and  $C$  be a nonempty, closed, and convex subset of  $H$ . Then  $p \in \Delta_A := \{x \in C : \langle A(x), z - x \rangle \geq 0, \forall z \in C\}$  if and only if  $p = P_C(I - \rho A)(p)$  for some  $\rho > 0$ , where  $P_C : H \rightarrow C$  denotes the metric projection of  $H$  onto  $C$ .

**Lemma 2.4.** (see [16, Corollary 2.42 and Lemma 2.35])

Let  $\{u_n\}$  be a sequence in  $H$ . Then the following statements hold:

- (1) If  $u_n \rightharpoonup u$  and  $\lim_{n \rightarrow \infty} \|u_n\| = \|u\|$ , then  $u_n \rightarrow u$ .
- (2) If  $u_n \rightharpoonup u$ , then  $\|u\| \leq \liminf_{n \rightarrow \infty} \|u_n\|$ .

**Lemma 2.5.** (see [16, Corollary 4.18])

Let  $C$  be a nonempty, closed, and convex subset of  $H$ , and let  $T : C \rightarrow C$  be nonexpansive. Let  $\{a_n\}$  be a sequence in  $C$  and let  $a \in H$ . Assume that  $a_n \rightharpoonup a$  and that  $a_n - T(a_n) \rightarrow 0$ . Then  $a \in \text{Fix}(T) := \{x \in C : T(x) = x\}$ .

## 3. Main results

For convenience in the sequel, we adopt the following notation throughout the paper:  $\mathcal{J}_N := \{1, 2, \dots, N\}$ ,  $\mathcal{J}_N^* := \{0, 1, \dots, N\}$ ,  $C = C_0$ ,  $H_0 = H$ ,  $A_0 = A$ , and, for each  $i \in \mathcal{J}_N^*$ , let  $I_i$  denote the identity mapping on  $H_i$ . Moreover, we set  $T_0 := I_0$  and  $\alpha_0 := \alpha$ , and assume that conditions (C1)–(C4) hold.

To find a solution of the SVI-MOS, we propose the following hybrid projection algorithm.

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### Inertial multistep hybrid projection algorithm (ALGO-IMHPA)

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**Step 0.** Choose sequences of real numbers  $\{\gamma_n\}$ ,  $\{\zeta_n\}$  and  $\{\mu_{j,n}\}$  for all  $j \in \{1, 2, \dots, p\}$ , and take

$\varrho \in (0, 2 \min_{i \in \mathcal{J}_N^*} \alpha_i)$ . Choose  $a_{-p}, a_{-p+1}, \dots, a_0 \in H$  arbitrarily, and let  $n := 0$ .

**Step 1.** Given  $a_{n-p}, a_{n-p+1}, \dots, a_n$ , compute

$$b_n = a_n + \sum_{j=1}^p \mu_{j,n} (a_{n-j+1} - a_{n-j}). \quad (1)$$

**Step 2.** For each  $i \in \mathcal{J}_N^*$ , compute  $c_{i,n}$  and  $d_{i,n}$  by

$$c_{i,n} = P_{C_i}(I_i - \varrho A_i)(T_i(b_n)), \quad (2)$$

$$d_{i,n} = b_n - \beta_{i,n} T_i^*(T_i(b_n) - c_{i,n}), \quad (3)$$

where  $\beta_{i,n} = \zeta_n \frac{\|T_i(b_n) - c_{i,n}\|^2}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma_n}$ .

**Step 3.** Choose an index  $i_n \in \mathcal{J}_N^*$  such that  $\|d_{i_n,n} - b_n\| = \max_{i \in \mathcal{J}_N^*} \|d_{i,n} - b_n\|$ , and set

$$d_n = d_{i_n,n}. \quad (4)$$

**Step 4.** Define two closed subsets  $\mathbf{U}_n$  and  $\mathbf{V}_n$  by

$$\mathbf{U}_n = \{z \in H : \|d_n - z\|^2 \leq \|b_n - z\|^2 - \Phi_n\}, \quad (5)$$

$$\mathbf{V}_n = \{z \in H : \langle z - a_n, a_0 - a_n \rangle \leq 0\}, \quad (6)$$

where  $\Phi_n = \zeta_n(1 - \zeta_n) \frac{\|T_{i_n}(b_n) - c_{i_n,n}\|^4}{\|T_{i_n}^*(T_{i_n}(b_n) - c_{i_n,n})\|^2 + \gamma_n}$ .

**Step 5.** Compute

$$a_{n+1} = P_{\mathbf{U}_n \cap \mathbf{V}_n}(a_0). \quad (7)$$

**Step 6.** Set  $n := n + 1$  and go to **Step 1**.

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The following theorem establishes the convergence of the sequence  $\{a_n\}$  generated by ALGO-IMHPA.

**Theorem 3.1.** Suppose that the following conditions hold:

(O1) For each  $j = 1, 2, \dots, p$ , the sequence  $\{\mu_{j,n}\}$  is bounded;

(O2)  $\{\zeta_n\} \subset [\zeta_*, \zeta^*] \subset (0, 1)$ .

(O3)  $\{\gamma_n\}$  is a positive sequence of real numbers and bounded from above by  $\gamma^*$ .

Then the sequence  $\{a_n\}$  generated by ALGO-IMHPA converges strongly to an element  $a_* \in \Delta$ , where  $a_* = P_\Delta(a_0)$ .

*Proof.* The proof is divided into several steps. Let  $s \in \Delta$  be arbitrary.

**Claim 1.** We have  $\Delta \subset \mathbf{U}_n \cap \mathbf{V}_n$  for all  $n \geq 0$ .

First, we rewrite  $\mathbf{U}_n$  and  $\mathbf{V}_n$  in equivalent half-space forms. Indeed, expanding  $\|d_n - z\|^2 \leq \|b_n - z\|^2 - \Phi_n$  yields  $\mathbf{U}_n = \{z \in H : \langle b_n - d_n, z \rangle \leq \frac{1}{2}(\|b_n\|^2 - \|d_n\|^2 - \Phi_n)\}$ , and  $\mathbf{V}_n$  can be written as  $\mathbf{V}_n = \{z \in H : \langle a_0 - a_n, z \rangle \leq \langle a_0 - a_n, a_n \rangle\}$ . Hence,  $\mathbf{U}_n$  and  $\mathbf{V}_n$  are closed half-spaces in  $H$ .

From (3), for each  $i \in \mathcal{J}_N^*$ , we have

$$\|d_{i,n} - s\|^2 = \|b_n - s - \beta_{i,n} T_i^*(T_i(b_n) - c_{i,n})\|^2$$

$$\begin{aligned}
&= \|b_n - s\|^2 + \beta_{i,n}^2 \|T_i^*(T_i(b_n) - c_{i,n})\|^2 - 2\beta_{i,n} \langle b_n - s, T_i^*(T_i(b_n) - c_{i,n}) \rangle \\
&= \|b_n - s\|^2 + \beta_{i,n}^2 \|T_i^*(T_i(b_n) - c_{i,n})\|^2 - 2\beta_{i,n} \langle T_i(b_n) - T_i(s), T_i(b_n) - c_{i,n} \rangle. \quad (8)
\end{aligned}$$

Moreover, we observe that

$$\langle T_i(b_n) - T_i(s), T_i(b_n) - c_{i,n} \rangle = \frac{\|T_i(b_n) - T_i(s)\|^2 + \|T_i(b_n) - c_{i,n}\|^2 - \|c_{i,n} - T_i(s)\|^2}{2}. \quad (9)$$

Since  $s \in \Delta$ , we have  $T_i(s) \in \Delta_{A_i}$ . This is equivalent to  $T_i(s) = P_{C_i}(I_i - \varrho A_i)(T_i(s))$  thanks to Lemma 2.3. Using (2),  $\varrho \in (0, 2 \min_{i \in \mathcal{J}_N^*} \alpha_i)$ , the nonexpansiveness of  $P_{C_i}$ , and the  $\alpha_i$ -inverse strong monotonicity of  $A_i$ , we obtain

$$\begin{aligned}
&\|c_{i,n} - T_i(s)\|^2 \\
&= \|P_{C_i}(I_i - \varrho A_i)(T_i(b_n)) - P_{C_i}(I_i - \varrho A_i)(T_i(s))\|^2 \\
&\leq \|(I_i - \varrho A_i)(T_i(b_n)) - (I_i - \varrho A_i)(T_i(s))\|^2 \\
&= \|T_i(b_n) - T_i(s)\|^2 + \varrho^2 \|A_i(T_i(b_n)) - A_i(T_i(s))\|^2 - 2\varrho \langle T_i(b_n) - T_i(s), A_i(T_i(b_n)) - A_i(T_i(s)) \rangle \\
&\leq \|T_i(b_n) - T_i(s)\|^2 + \varrho^2 \|A_i(T_i(b_n)) - A_i(T_i(s))\|^2 - 2\varrho \alpha_i \|A_i(T_i(b_n)) - A_i(T_i(s))\|^2 \\
&= \|T_i(b_n) - T_i(s)\|^2 - \varrho(2\alpha_i - \varrho) \|A_i(T_i(b_n)) - A_i(T_i(s))\|^2 \leq \|T_i(b_n) - T_i(s)\|^2. \quad (10)
\end{aligned}$$

Combining this with (9) gives

$$\langle T_i(b_n) - T_i(s), T_i(b_n) - c_{i,n} \rangle \geq \frac{1}{2} \|T_i(b_n) - c_{i,n}\|^2. \quad (11)$$

Substituting (11) into (8) and using the definition of  $\beta_{i,n}$ , we deduce

$$\begin{aligned}
\|d_{i,n} - s\|^2 &\leq \|b_n - s\|^2 + \beta_{i,n}^2 \|T_i^*(T_i(b_n) - c_{i,n})\|^2 - \beta_{i,n} \|T_i(b_n) - c_{i,n}\|^2 \\
&\leq \|b_n - s\|^2 + \zeta_n^2 \frac{\|T_i(b_n) - c_{i,n}\|^4}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma_n} - \zeta_n \frac{\|T_i(b_n) - c_{i,n}\|^4}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma_n} \\
&= \|b_n - s\|^2 - \zeta_n(1 - \zeta_n) \frac{\|T_i(b_n) - c_{i,n}\|^4}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma_n}, \quad (12)
\end{aligned}$$

for all  $i \in \mathcal{J}_N^*$ . By the choice of  $i_n$  in Step 3 and the definition (4), it follows that

$$\|d_n - s\|^2 \leq \|b_n - s\|^2 - \zeta_n(1 - \zeta_n) \frac{\|T_{i_n}(b_n) - c_{i_n,n}\|^4}{\|T_{i_n}^*(T_{i_n}(b_n) - c_{i_n,n})\|^2 + \gamma_n} = \|b_n - s\|^2 - \Phi_n. \quad (13)$$

It follows from (5) and (13) that  $s \in \mathbf{U}_n$  for all  $n \geq 0$ , that is,  $\Delta \subset \mathbf{U}_n$  for all  $n \geq 0$ .

Next, we show that  $\Delta \subset \mathbf{V}_n$  for all  $n \geq 0$  by induction. Clearly,  $\mathbf{V}_0 = H$ , so  $\Delta \subset \mathbf{V}_0$ . Assume that  $\Delta \subset \mathbf{V}_n$  for some  $n \geq 0$ . Since  $a_{n+1} = P_{\mathbf{U}_n \cap \mathbf{V}_n}(a_0)$ , the characterization of metric projections onto closed convex sets (Lemma 2.2 (1)) yields

$$\langle a_0 - a_{n+1}, z - a_{n+1} \rangle \leq 0, \quad \forall z \in \mathbf{U}_n \cap \mathbf{V}_n.$$

Taking  $z = s \in \Delta \subset \mathbf{U}_n \cap \mathbf{V}_n$ , we get  $\langle a_0 - a_{n+1}, s - a_{n+1} \rangle \leq 0$ , which implies  $s \in \mathbf{V}_{n+1}$ . Hence,  $\Delta \subset \mathbf{V}_n$  for all  $n \geq 0$ . Consequently,  $\Delta \subset \mathbf{U}_n \cap \mathbf{V}_n$  for all  $n \geq 0$ . In particular,  $\mathbf{U}_n \cap \mathbf{V}_n \neq \emptyset$ , so  $\{a_n\}$  is well defined.

**Claim 2.** The sequence  $\{a_n\}$  is bounded.

In view of Lemma 2.2 (1) and (6), we find that  $a_n = P_{V_n}(a_0)$ . Since  $a_* = P_\Delta(a_0) \in \Delta \subset V_n$ , we have

$$\|a_n - a_0\| \leq \|a_* - a_0\|, \quad \forall n \geq 0, \quad (14)$$

which shows that the sequence  $\{a_n\}$  is indeed bounded, as claimed. It also follows from (C3), (1), condition (O1), (10), (12), and (13) that the sequences  $\{b_n\}$ ,  $\{T_i(b_n)\}$ ,  $\{d_n\}$ ,  $\{c_{i,n}\}$  and  $\{d_{i,n}\}$  ( $i \in \mathcal{J}_N^*$ ) are bounded too.

**Claim 3.** The following limits hold:

$$\lim_{n \rightarrow \infty} \|a_{n+1} - a_n\| = 0, \quad (15)$$

$$\lim_{n \rightarrow \infty} \|a_{n+1} - b_n\| = 0, \quad (16)$$

$$\lim_{n \rightarrow \infty} \|b_n - d_n\| = 0. \quad (17)$$

Using Lemma 2.2 (2), the fact that  $a_n = P_{V_n}(a_0)$ , and (7), we obtain

$$0 \leq \|a_{n+1} - a_n\|^2 \leq \|a_{n+1} - a_0\|^2 - \|a_n - a_0\|^2. \quad (18)$$

This implies that  $\|a_{n+1} - a_0\| \geq \|a_n - a_0\|$ , so the sequence  $\{\|a_n - a_0\|\}$  is nondecreasing. By Claim 2, it is bounded, and thus, the finite limit  $l = \lim_{n \rightarrow \infty} \|a_n - a_0\|$  exists. Consequently, letting  $n \rightarrow \infty$  in (18) yields  $\lim_{n \rightarrow \infty} \|a_{n+1} - a_n\| = 0$ , which proves (15).

Next, it follows from (1) that

$$\|a_{n+1} - b_n\| \leq \|a_{n+1} - a_n\| + \sum_{j=1}^p |\mu_{j,n}| \|a_{n-j+1} - a_{n-j}\|,$$

and hence, by (O1), we obtain (16).

Furthermore, since  $a_{n+1} \in U_n$  and  $\Phi_n \geq 0$ , it follows from (5) that  $\|a_{n+1} - d_n\|^2 \leq \|a_{n+1} - b_n\|^2$ . Combining this with (16), we infer that  $\lim_{n \rightarrow \infty} \|a_{n+1} - d_n\| = 0$ . Using the estimate  $\|b_n - d_n\| \leq \|a_{n+1} - b_n\| + \|a_{n+1} - d_n\|$ , we arrive at the limit (17), as claimed.

**Claim 4.** The sequence  $\{a_n\}$  converges strongly to  $a_* = P_\Delta(a_0)$ .

It follows from condition (O2), (O3), (4) and (12) that

$$\begin{aligned} 0 &\leq \zeta_*(1 - \zeta^*) \frac{\|T_i(b_n) - c_{i,n}\|^4}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma^*} \leq \zeta_n(1 - \zeta_n) \frac{\|T_i(b_n) - c_{i,n}\|^4}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma_n} \\ &\leq \|b_n - s\|^2 - \|d_{i,n} - s\|^2 \leq M_0(\|b_n - s\| - \|d_{i,n} - s\|) \\ &\leq M_0\|b_n - d_{i,n}\| \leq M_0\|b_n - d_n\|, \end{aligned} \quad (19)$$

for all  $i \in \mathcal{J}_N^*$ , where  $M_0 = \sup_n \{\|b_n - s\| + \|d_n - s\|\} < \infty$ . Using (O2), (17) and (19), we obtain

$$\lim_{n \rightarrow \infty} \frac{\|T_i(b_n) - c_{i,n}\|^4}{\|T_i^*(T_i(b_n) - c_{i,n})\|^2 + \gamma^*} = 0, \quad \forall i \in \mathcal{J}_N^*.$$

This implies that

$$\lim_{n \rightarrow \infty} \|T_i(b_n) - c_{i,n}\| = 0, \quad \forall i \in \mathcal{J}_N^*, \quad (20)$$

thanks to the boundedness of the  $\{T_i(b_n) - c_{i,n}\}$ .

Next, since  $\{a_n\}$  is a bounded sequence, there exists a subsequence  $\{a_{n_k}\}$  of  $\{a_n\}$  which converges weakly to some point  $x^* \in H$ , that is,  $a_{n_k} \rightharpoonup x^*$ . Moreover, (15) and (16) imply that  $\lim_{n \rightarrow \infty} \|a_n - b_n\| = 0$ , and hence  $b_{n_k} \rightharpoonup x^*$ . Since each  $T_i$  is a bounded linear operator, we also have

$$T_i(b_{n_k}) \rightharpoonup T_i(x^*), \quad \forall i \in \mathcal{J}_N^*. \quad (21)$$

Furthermore, from (2) and (20), we obtain

$$\lim_{k \rightarrow \infty} \|T_i(b_{n_k}) - P_{C_i}(I_i - \varrho A_i)(T_i(b_{n_k}))\| = \lim_{k \rightarrow \infty} \|T_i(b_{n_k}) - c_{i,n_k}\| = 0, \quad \forall i \in \mathcal{J}_N^*. \quad (22)$$

By Lemma 2.1,  $P_{C_i}(I_i - \varrho A_i)$  is nonexpansive for  $\varrho \in (0, 2 \min_{i \in \mathcal{J}_N^*} \alpha_i)$ . Therefore, using (21), (22), and applying Lemma 2.5, it follows that  $T_i(x^*) \in \text{Fix}(P_{C_i}(I_i - \varrho A_i))$  for all  $i \in \mathcal{J}_N^*$ , is equivalent to  $T_i(x^*) = P_{C_i}(I_i - \varrho A_i)(T_i(x^*))$  for all  $i \in \mathcal{J}_N^*$ . Consequently, by Lemma 2.3, we obtain  $T_i(x^*) \in \Delta_{A_i}$  for all  $i \in \mathcal{J}_N^*$ . This means that  $x^* \in \Delta$ .

Finally, since  $a_* = P_\Delta(a_0)$  and  $x^* \in \Delta$ , we have

$$\|a_* - a_0\| \leq \|x^* - a_0\| \leq \liminf_{k \rightarrow \infty} \|a_{n_k} - a_0\| \leq \limsup_{k \rightarrow \infty} \|a_{n_k} - a_0\| \leq \|a_* - a_0\|,$$

thanks to (14) and Lemma 2.4 (2). Consequently,  $\|x^* - a_0\| = \|a_* - a_0\|$ , and by the uniqueness of the metric projection, we conclude that  $x^* = a_*$ . Moreover,  $\lim_{k \rightarrow \infty} \|a_{n_k} - a_0\| = \|a_* - a_0\|$ . In view of Lemma 2.4 (1), it follows that  $a_{n_k} \rightarrow a_*$ . Since every weakly convergent subsequence of  $\{a_n\}$  has the same strong limit  $a_*$ , we deduce that  $a_n \rightarrow a_*$ .  $\square$

**Remark 3.2.** It can be readily verified that the SVI-MOS becomes the split feasibility problem with multiple output sets (SFP-MOS) by choosing  $A = 0$ ,  $A_i = 0$ ,  $\forall i \in \mathcal{J}_N$ . More precisely, the problem then consists of finding an element  $x_* \in C$  such that  $T_i(x_*) \in C_i$  for all  $i \in \mathcal{J}_N$ . Consequently, the algorithm ALGO-IMHPA is applicable for computing an approximate solution of the SFP-MOS, provided that (2) is replaced with  $c_{i,n} = P_{C_i}(T_i(b_n))$  for each  $i \in \mathcal{J}_N^*$ .

**Remark 3.3.** Let  $f_i : C_i \rightarrow \mathbb{R}$  be a convex Fréchet differentiable function whose Fréchet derivative  $\nabla f_i$  is  $1/\alpha_i$ -Lipschitz continuous, where  $\alpha_i > 0$  and  $i \in \mathcal{J}_N^*$ . Denote by  $\arg \min_{x \in C_i} f_i(x)$ , the set of all minimizers of  $f_i$  over  $C_i$ . Since  $\nabla f_i$  is  $1/\alpha_i$ -Lipschitz continuous, it follows that  $\nabla f_i$  is an  $\alpha_i$ -inverse strongly monotone operator (see, for example, [16, Corollary 18.16]). Moreover, a point  $x_* \in C_i$  belongs to  $\arg \min_{x \in C_i} f_i(x)$  if and only if  $\langle \nabla f_i(x_*), z - x_* \rangle \geq 0$ , for all  $z \in C_i$  (see, for instance, [16, Proposition 26.5]). Therefore, by taking  $A = \nabla f_0$ ,  $A_i = \nabla f_i$ ,  $\forall i \in \mathcal{J}_N$ , the SVI-MOS reduces to the SCMPP-MOS. In other words, one needs to seek a point  $x_* \in \arg \min_{x \in C_0} f_0(x)$  such that  $T_i(x_*) \in \arg \min_{x \in C_i} f_i(x)$ , for all  $i \in \mathcal{J}_N^*$ . Accordingly, the algorithm ALGO-IMHPA can be employed to find an approximate solution of the SCMPP-MOS, provided that (2) is replaced with  $c_{i,n} = P_{C_i}(I_i - \varrho \nabla f_i)(T_i(b_n))$  for each  $i \in \mathcal{J}_N^*$ .

**Remark 3.4.** Let  $B_i : C_i \rightarrow C_i$  be a nonexpansive mapping for each  $i \in \mathcal{J}_N^*$ . Denote by  $\text{Fix}(B_i) = \{x \in C_i : B_i(x) = x\}$ , the fixed point set of  $B_i$ . According to a basic result, the operator  $I_i - B_i$  is a  $1/2$ -inverse strongly monotone mapping (see, for example, [9, Section 4]). Additionally, it can be verified that  $x_* \in \text{Fix}(B_i)$  whence  $x_* \in C_i$  and for every  $z \in C_i$ , we have  $\langle (I_i - B_i)(x_*), z - x_* \rangle \geq 0$ . This shows that the SCFPP-MOS can be embedded into the SVI-MOS framework by choosing  $A = I_0 - B_0$ ,  $A_i = I_i - B_i$ ,  $\forall i \in \mathcal{J}_N^*$ . Under this choice, the task is to determine a point  $x_* \in \text{Fix}(B_0)$  satisfying  $T_i(x_*) \in \text{Fix}(B_i)$ , for all  $i \in \mathcal{J}_N$ . Hence, the iterative procedure ALGO-IMHPA remains applicable to this setting, with (2) taking the form  $c_{i,n} = P_{C_i}((1 - \varrho)T_i(b_n) + \varrho B_i(T_i(b_n)))$  for every  $i \in \mathcal{J}_N^*$ .

#### 4. Conclusion

In this paper, we have proposed a new algorithm for the split variational inequality problem with multiple output sets in real Hilbert spaces. The strong convergence of the generated sequence has been established under mild assumptions. Future work may be devoted to extending the proposed method to more general frameworks, including problems with nonmonotone operators, and to investigating its potential applications in signal processing and image reconstruction.

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