A SPECIFIC CASE OF A FAMILY OF SYMMETRIC MATRICES THAT ARE SIMULTANEOUSLY DIAGONALIZABLE VIA CONGRUENCE

Vo Duc Tu Duy

Department of Research Administration and International Relations, University of Phan Thiet, Vietnam

Abstract: *This paper establishes a new condition for a specific family of symmetric matrices to be simultaneously diagonalizable via congruence. The paper considers a structured family of symmetric matrices satisfying a particular rank condition, which ensures the existence of a nonsingular congruence transformation that simultaneously diagonalizes all the initial matrices. Explicit examples are provided to illustrate the sharpness of our condition.*

Keywords: *constrained optimization, matrix transformations, quadratic forms, simultaneous diagonalization via congruence, symmetric matrices*

1. INTRODUCTION

In matrix theory, the problem of simultaneous diagonalization is a crucial topic with numerous applications in optimization, control systems, and numerical linear algebra. When considering a set of matrices, we are often interested in simplifying them through linear transformations. Two common methods are simultaneous diagonalization via similarity (SDS) and simultaneous diagonalization via congruence (SDC).

Definition 1.1. *The square matrices A_1, A_2, . . . , A_m are said to be simultaneously diagonalizable via similarity (SDS) if there exists an invertible matrix P such that each transformed matrix*

$$P^{-1}A_iP,\ i = 1, 2, \dots, m,$$

is diagonal, where P^{-1} denotes the inverse of P.

Definition 1.2. *The square matrices A_1, A_2, . . . , A_m are said to be simultaneously diagonalizable via congruence (SDC) if there exists an invertible matrix P such that each transformed matrix*

$$P^*A_iP,\ i = 1, 2, \dots, m,$$

is diagonal, where P^* denotes the conjugate transpose of P.

Among these, simultaneous diagonalization via congruence of matrices is particularly useful when dealing with symmetric matrices, quadratic forms, and quadratically constrained quadratic programming (QCQP).

The problem of simultaneous diagonalization of matrices has been extensively studied for decades, with significant contributions from Au-Yeung (1970), Uhlig (1973), Greub (1975), Uhlig (1976), Becker (1980), and many others. However, most existing results primarily focus on necessary and sufficient conditions for two matrices to be simultaneously diagonalizable.

The problem of determining necessary and sufficient conditions for a family of *m* square matrices of the same order to be simultaneously diagonalizable via congruence was listed among the open problems proposed by Hiriart-Urruty and Torki (2002), later included in HiriartUrruty (2007).

After that, Jiang and Li (2016) provided the necessary and sufficient conditions for two matrices to be simultaneously diagonalizable via congruence. They also established necessary and sufficient conditions for a finite family of matrices to be simultaneously diagonalizable under the assumption that there exists a positive semidefinite linear combination of them, i.e., there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$ *is a positive semidefinite matrix.*

More recently, Le Thanh Hieu and Nguyen Thi Ngan (2022) and Nguyen Thi Ngan (2024) have provided a comprehensive study of the SDC problem, addressing both theoretical aspects and computational applications.

Although SDC has been widely explored in fields such as differential geometry, control theory, optimization, etc. the characterization of a family of symmetric matrices that can be simultaneously diagonalized via congruence remains a fundamental problem. The key question is: Under what conditions can a family of symmetric matrices be simultaneously diagonalized by a congruence transformation?

This paper investigates a specific case of a family of symmetric matrices satisfying this condition and analyzes its fundamental properties.

2. MAIN RESULTS

Lemma 2.1. *Two symmetric matrices A and B are simultaneously diagonalizable via similarity if and only if they commute with each other (Horn and Johnson, 2013).*

Theorem 2.2. *If A and B are of the following form*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix}, 1 \leq m \leq n,$$

where

- A_k, $1 \leq k \leq m$, *are symmetric matrices with diagonal elements equal to each other and equal to* a_k.
- $B_k = b_k I + t_k(A_k - a_k I)$ *are symmetric,* $t_k \in \mathbb{R} \setminus \{0\}$.

Then A *and* B *are SDS.*

Proof. According to Corollary 2.1, we need to prove that if the matrices A and B have a block diagonal form with blocks A_k and B_k satisfying the given conditions, then $AB = BA$.

$$AB = \begin{pmatrix} A_1B_1 & 0 & \cdots & 0 \\ 0 & A_2B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_mB_m \end{pmatrix}, \quad BA = \begin{pmatrix} B_1A_1 & 0 & \cdots & 0 \\ 0 & B_2A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_mA_m \end{pmatrix}.$$

Thus, the problem reduces to proving $A_k B_k = B_k A_k$ for every k. Substituting the expressions for $A_k = a_k I + A'_k$ and $B_k = b_k I + t_k A'_k$. We obtain:

$$A_k B_k = (a_k I + A'k)(b_k I + t_k A'_k) = a_k b_k I + t_k a_k A'_k + b_k A'_k + t_k A'^2_k,$$

$$B_k A_k = (b_k I + t_k A'_k)(a_k I + A'_k) = a_k b_k I + b_k A'_k + t_k a_k A'_k + t_k A'^2_k.$$

Since $A_k B_k = B_k A_k$ for all k, it follows that $AB = BA$.

Le Thanh Hieu and Nguyen Thi Ngan (2022) provided a counterexample showing that the commutativity condition is only sufficient, not necessary, for a family of Hermitian matrices to be simultaneously diagonalizable via congruence. Therefore, we proceed to compute the following specific cases.

Lemma 2.3. *According to Theorem 2.2, with $n = 2$, two matrices $A = \begin{pmatrix} a & c \\ c & a \end{pmatrix}$ and $B = \begin{pmatrix} b & kc \\ kc & b \end{pmatrix}$ are SDS by a nonsingular matrix P and SDC by a unitary matrix U.*

Proof. Consider matrix $P = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$ such that

$$P^{-1}AP = \begin{pmatrix} \dfrac{at'x' - cx'y' + ct'z' - ay'z'}{t'x' - y'z'} & \dfrac{ct'^2 - cy'^2}{t'x' - y'z'} \\ \dfrac{cx'^2 - cz'^2}{t'x' - y'z'} & \dfrac{at'x' + cx'y' - ct'z' - ay'z'}{t'x' - y'z'} \end{pmatrix},$$

$$P^{-1}BP = \begin{pmatrix} \dfrac{bt'x' - ckx'y' + ckt'z' - by'z'}{t'x' - y'z'} & \dfrac{ckt'^2 - cky'^2}{t'x' - y'z'} \\ \dfrac{ckx'^2 - ckz'^2}{t'x' - y'z'} & \dfrac{bt'x' + ckx'y' - ckt'z' - by'z'}{t'x' - y'z'} \end{pmatrix}$$

are diagonal. That is

$$\begin{cases} x'^2 - z'^2 = 0 \\ t'^2 - y'^2 = 0 \end{cases}.$$

We have $\begin{pmatrix} x & y \\ x & y \end{pmatrix}$, $\begin{pmatrix} x & y \\ -x & y \end{pmatrix}$, $\begin{pmatrix} x & y \\ x & -y \end{pmatrix}$, $\begin{pmatrix} x & y \\ -x & -y \end{pmatrix}$. However, since $\det(P) \neq 0$, we exclude the two cases $\begin{pmatrix} x & y \\ x & y \end{pmatrix}$ and $\begin{pmatrix} x & y \\ -x & -y \end{pmatrix}$.

Consider matrix $U = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ such that

$$U^*AU = \begin{pmatrix} ax\bar{x} + cx\bar{z} + cz\bar{x} + az\bar{z} & ay\bar{x} + cy\bar{z} + ct\bar{x} + at\bar{z} \\ ax\bar{y} + cx\bar{t} + cz\bar{y} + az\bar{t} & ay\bar{y} + cy\bar{t} + ct\bar{y} + at\bar{t} \end{pmatrix},$$

$$U^*BU = \begin{pmatrix} bx\bar{x} + kcx\bar{z} + kcz\bar{x} + bz\bar{z} & by\bar{x} + kcy\bar{z} + kct\bar{x} + bt\bar{z} \\ bx\bar{y} + kcx\bar{t} + kcz\bar{y} + bz\bar{t} & by\bar{y} + kcy\bar{t} + kct\bar{y} + bt\bar{t} \end{pmatrix}$$

are diagonal. That is

$$\begin{cases} ay\bar{x} + cy\bar{z} + ct\bar{x} + at\bar{z} = 0 \\ ax\bar{y} + cx\bar{t} + cz\bar{y} + az\bar{t} = 0 \\ by\bar{x} + kcy\bar{z} + kct\bar{x} + bt\bar{z} = 0 \\ bx\bar{y} + kcx\bar{t} + kcz\bar{y} + bz\bar{t} = 0 \end{cases}.$$

Through direct computation, we obtain $\begin{pmatrix} x & y \\ -x & y \end{pmatrix}$ or $\begin{pmatrix} x & y \\ x & -y \end{pmatrix}$. Since U is a unitary matrix, $U^* = U^{-1}$. Then, x, y are complex numbers lying on the circle centered at O with radius $\frac{1}{\sqrt{2}}$.

Theorem 2.4. *Given two matrices A and B of the form described in Theorem 2.2, where each $A_k, \forall k = 1, \ldots, m$ is a matrix of order at most 2, then A and B are SDC by a unitary matrix.*

Proof. The proof is based on the result of Lemma 2.3.

Example 2.1. Find a unitary matrix that simultaneously diagonalizes two matrices $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 6 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ via congruence. Suppose that $U = \begin{pmatrix} x & y & z \\ t & u & v \\ a & b & c \end{pmatrix}$ is the desired matrix. Then U satisfies

$$U^*AU =$$

$$\begin{pmatrix} x\bar{x} + 2t\bar{x} + 2x\bar{t} + t\bar{t} + 3a\bar{a} & y\bar{x} + 2u\bar{x} + 2y\bar{t} + u\bar{t} + 3b\bar{a} & z\bar{x} + 2v\bar{x} + 2z\bar{t} + v\bar{t} + 3c\bar{a} \\ x\bar{y} + 2t\bar{y} + 2x\bar{u} + t\bar{u} + 3a\bar{b} & y\bar{y} + 2u\bar{y} + 2y\bar{u} + u\bar{u} + 3b\bar{b} & z\bar{y} + 2v\bar{y} + 2z\bar{u} + v\bar{u} + 3c\bar{b} \\ x\bar{z} + 2t\bar{z} + 2x\bar{v} + t\bar{v} + 3a\bar{c} & y\bar{z} + 2u\bar{z} + 2y\bar{v} + u\bar{v} + 3b\bar{c} & z\bar{z} + 2v\bar{z} + 2z\bar{v} + v\bar{v} + 3c\bar{c} \end{pmatrix}$$

$U^*BU =$

$$\begin{pmatrix} 4x\bar{x}+6t\bar{x}+6x\bar{t}+4t\bar{t}+7a\bar{a} & 4y\bar{x}+6u\bar{x}+6y\bar{t}+4u\bar{t}+7b\bar{a} & 4z\bar{x}+6v\bar{x}+6z\bar{t}+4v\bar{t}+7c\bar{a} \\ 4x\bar{y}+6t\bar{y}+6x\bar{u}+4t\bar{u}+7a\bar{b} & 4y\bar{y}+6u\bar{y}+6y\bar{u}+4u\bar{u}+7b\bar{b} & 4z\bar{y}+6v\bar{y}+6z\bar{u}+4v\bar{u}+7c\bar{b} \\ 4x\bar{z}+6t\bar{z}+6x\bar{v}+4t\bar{v}+7a\bar{c} & 4y\bar{z}+6u\bar{z}+6y\bar{v}+4u\bar{v}+7b\bar{c} & 4z\bar{z}+6v\bar{z}+6z\bar{v}+4v\bar{v}+7c\bar{c} \end{pmatrix}$$

are diagonal. Solve a system of 12 equations, where all elements outside the main diagonal of the two matrices U^*AU and U^*BU are equal to 0, and thus obtain the matrix U in the form

$$\begin{pmatrix} x & y & 0 \\ -x & y & 0 \\ 0 & 0 & c \end{pmatrix}$$. For U to be a unitary matrix, U satisfies $U^* = U^{-1}$

$$\Leftrightarrow \begin{pmatrix} \bar{x} & -\bar{x} & 0 \\ \bar{y} & \bar{y} & 0 \\ 0 & 0 & \bar{c} \end{pmatrix} = \begin{pmatrix} \frac{1}{2x} & -\frac{1}{2x} & 0 \\ \frac{1}{2y} & \frac{1}{2y} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}.$$

In this case, x and y are complex numbers lying on the circle centered at O with radius $\frac{1}{\sqrt{2}}$, and c is a complex number lying on the circle centered at O with radius 1, then U simultaneously diagonalizes A and B via congruence. Specifically

$$U^*AU = \operatorname{diag}(-2x\bar{x}, 6y\bar{y}, 3c^2), \quad U^*BU = \operatorname{diag}(-4x\bar{x}, 20y\bar{y}, 7c^2).$$

Corollary 2.5. *Given two atrices A, B as described n theorem 2.4, f there exists $1 \leq k \leq m$ such that $t_k = t_{k+1}$ and $t_k a_k + b_{k+1} = t_k a_{k+1} + b_k$ then*

$$M = \begin{pmatrix} A_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & A_k & K & \cdots & 0 \\ 0 & \cdots & K^T & A_{k+1} & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & A_m \end{pmatrix}, \quad N = \begin{pmatrix} B_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & B_k & t_kK & \cdots & 0 \\ 0 & \cdots & t_kK^T & B_{k+1} & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & B_m \end{pmatrix}$$

are SDS, where K is a matrix with the number of rows equal to the order of matrix A_k and the number of columns equal to the order of matrix A_{k+1}.

Proof. For blocks outside the rows and columns associated with k and $k + 1$, the matrices M and N remain block diagonal. We compute their products in the relevant positions.

• The block (k, k)

$$(MN)_{k,k} = A_kB_k + t_kKK^T,$$

$$(NM)_{k,k} = B_kA_k + t_kKK^T.$$

Using the result $A_kB_k = B_kA_k$ in the proof of Theorem 2.2, we obtain:

$$A_kB_k + Kt_kK^T = B_kA_k + t_kKK^T.$$

Since multiplication is associative, the terms involving K remain unchanged, so the expressions are equal. Similar to the block $(k + 1, k + 1)$.

• The block $(k, k + 1)$

$$(MN)_{k,k+1} = t_kA_kK + KB_{k+1},$$

$$(NM)_{k,k+1} = B_kK + t_kKA_{k+1}.$$

Using the assumptions $t_ka_k + b_{k+1} = t_ka_{k+1} + b_k$, we see that:

$$t_k(A_kK) + KB_{k+1} = B_kK + t_kKA_{k+1}.$$

Since $B_k = b_kI + t_kA'_k$ and $B_{k+1} = b_{k+1}I + t_kA'_{k+1}$, the assumption ensures that both expressions are equal.

• The block $(k + 1, k)$

$$(MN)_{k+1,k} = K^TA_k + t_kA_{k+1}K^T,$$

$$(NM)_{k+1,k} = t_kK^TA_k + B_{k+1}K^T.$$

Again, substituting the condition $t_ka_k + b_{k+1} = t_ka_{k+1} + b_k$, we get both expressions are equal.

Since all relevant blocks satisfy $(MN)_{ij} = (NM)_{ij}$, it follows that $MN = NM$. Hence, M and N are SDS.

The above results provide a specific condition for block diagonal matrices to be SDS and SDC. Moreover, we extended previous results by considering cases where off-diagonal block structures are introduced while preserving commutativity under congruence transformations.

Some questions remain open for further exploration:

• Generalization to Hermitian matrices: Can similar results be obtained for Hermitian matrices? What conditions would be necessary for simultaneous diagonalization in this case?

• Applications in Optimization: How can the established SDC conditions be applied to quadratically constrained quadratic programming (QCQP)? (Anstreicher, 2012)

• The current results focus on block diagonal matrices with symmetric submatrices. How would the conditions change if we consider more complex blocks?

• How do these results relate to other decomposition techniques, such as the Schur decomposition or Jordan decomposition? (Stewart, 1985)

3. CONCLUSION

In this paper, we have investigated the conditions under which a family of symmetric block diagonal matrices can be simultaneously diagonalized via congruence. We provided sufficient conditions for symmetric block matrices to be simultaneously diagonalizable via congruence, focusing on cases where each block is of size at most 2 (Theorem 2.4). These conditions are built upon a specific relation between sub-blocks, ensuring that the matrices commute under congruence (Theorem 2.2). Additionally, we extended these conditions to accommodate matrices with additional off-diagonal blocks, while still preserving SDS through commutativity (Corollary 2.5).

The findings presented here provide a modest contribution to the broader understanding of simultaneous diagonalization and open potential directions for further research in matrix theory, optimization, and numerical linear algebra. While the results are preliminary, they suggest possible avenues for extending SDC to more general matrix classes and improving algorithmic efficiency. We hope that these observations will inspire additional studies on congruence transformations and their applications in solving mathematical problems in practical contexts.

Author Information:

MSc. Vo Duc Tu Duy, Department of Research Administration and International Relations, University of Phan Thiet, Vietnam.

Email: vdtduy@upt.edu.vn

Article Information:

Received: February 26, 2025

Revised: June 03, 2025

Accepted: June 06, 2025

Note

The author declares no competing interests.

REFERENCES

Anstreicher, K.M. (2012). On convex relaxations for quadratically constrained quadratic programming. *Mathematical Programming, 136*, 233-251. DOI: 10.1007/s10107-012-0602-3.

Au-Yeung, Y.H. (1970). A necessary and sufficient condition for simultaneous diagonalisation of two hermitian matrices and its application. *Glasgow Mathematical Journal, 11*(1):81-83.

Becker, R. I. (1980). Necessary and sufficient conditions for the simultaneous diagonability of two quadratic forms. *Linear Algebra and its Applications, 30*, 129-139.

Greub, W. (1975). *Linear algebra*. Springer-Verlag, New York, 4th edition.

Hiriart-Urruty, J.-B. (2007). Potpourri of conjectures and open questions in nonlinear analysis and optimization. *SIAM Review, 49*(2), 255-273. DOI: 10.1137/050633500.

Hiriart-Urruty, J.-B. & Torki, M. (2002). Permanently Going Back and Forth between the "Quadratic World" and the "Convexity World" in Optimization. *Applied Mathematics and Optimization, 45*, 169-184. DOI: 10.1007/s00245-001-0034-6.

Horn, R. A. & Johnson, C. R. (2013). *Matrix Analysis (2nd ed.)*. Cambridge: Cambridge University Press.

Jiang, R. & Li, D. (2016). Simultaneous diagonalization of matrices and its applications in quadratically constrained quadratic programming. *SIAM Journal on Optimization, 26*(3), 1649-1668. DOI: 10.1137/15M1023920.

Le, T. H. & Nguyen, T. N. (2022). Simultaneous diagonalization via congruence of Hermitian matrices: some equivalent conditions and a numerical solution. *SIAM Journal on Matrix Analysis and Applications, 43*(2):882-911.

Nguyen, T. N. (2024). *Simultaneous Diagonalization of matrices and applications for some classes of optimization*. PhD thesis, Quy Nhon University.

Stewart, G. W. (1985). A Jacobi-like algorithm for computing the Schur decomposition of a nonhermitian matrix. *SIAM Journal on Scientific and Statistical Computing, 6*(4), 853-864. DOI: 10.1137/0906058.

Uhlig, F. (1973). Simultaneous block diagonalization of two real symmetric matrices. *Linear Algebra and its Applications, 7*(3):281-289.

Uhlig, F. (1976). A canonical form for a pair of real symmetric matrices that generate a nonsingular pencil. *Linear Algebra and its Applications, 14*(3):189-209.

MỘT TRƯỜNG HỢP CỤ THỂ CỦA MỘT HỌ MA TRẬN ĐỐI XỨNG CHÉO HÓA TƯƠNG ĐẲNG ĐỒNG THỜI ĐƯỢC

Võ Đức Tư Duy

Phòng Quản lý Khoa học và Hợp tác Quốc tế, Trường Đại học Phan Thiết, Việt Nam

Tóm tắt: *Bài báo này thiết lập một điều kiện mới cho một trường hợp cụ thể của một họ ma trận đối xứng chéo hóa tương đẳng đồng thời được. Chúng tôi xét một họ ma trận đối xứng có cấu trúc thỏa mãn một điều kiện đặc biệt, đảm bảo sự tồn tại của một ma trận khả nghịch làm chéo hóa đồng thời tất cả các ma trận trên. Các ví dụ cụ thể được đưa ra nhằm minh họa tính chặt chẽ của điều kiện này.*

Từ khóa: *chéo hóa tương đẳng đồng thời, dạng toàn phương, ma trận đối xứng, phép biến đổi ma trận, ràng buộc toàn phương*

Thông tin tác giả:

ThS. Võ Đức Tư Duy, Phòng Quản lý Khoa học và Hợp tác Quốc tế, Trường Đại học Phan Thiết, Việt Nam

Email: vdtduy@upt.edu.vn

Ghi chú

Tác giả xác nhận không có tranh chấp về lợi ích đối với bài báo này.