LAX - FRIEDRICHS REGULARIZATION DIFFERENCE ALGORITHM FOR TRAFFIC DENSITY FORECASTING PROBLEM WITH DELAY

Nguyen Dinh Dung

University of Information and Communication Technology, Thai Nguyen University, Vietnam

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****Correspondence:***
nddung@ictu.edu.vn
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ABSTRACT

In recent years, traffic density forecasting has been playing an important role in developing and improving the performance of intelligent traffic systems. Traffic density forecasting to optimize traffic management, urban management of vehicular traffic have the ability to coordinate traffic, optimize traffic light signals and apply intelligent regulation based on forecasts, thereby improving the handling of the road segment and reducing travel time. Therefore, the construction of predictive algorithms along with integration into traffic management systems is essential to promote the sustainable development of intelligent transportation systems in the future. In this research, an algorithm has been developed to predict traffic density on the delayed Lighthill - Whitham - Richards (LWR) model, in which a regularization difference method has been proposed as the basis for the algorithm. This article mainly focus on building an algorithm and performing experimental calculations to verify the correctness of the algorithm on a mathematical model.

Keywords: Traffic flow models; Lax - Fedrichs difference; delayed LWR model; regularization difference; traffic density

1. Introduction

The problem of determining the traffic density was first known in 1935 by the American scientist Greenshields. In 2008, Abdul Salaam [1] developed a mathematical model of the problem as follows: Assume, the road surface is rectangular with area S= (length of road) x (width), due to street divided into lanes with separation (hard or soft), vehicles always stay in one lane, passing each other does not affect the average speed in a segmen Δx, driving direction is fixed for each street.

Traffic is described in terms of macroscopic variables such as density $u(x,t)$ is the traffic density at the position x on the road section and at the time $\frac{1}{2}$, this quantity has the unit of measure as the number of vehicles per unit of distance. $J(x,t)$ is the vehicle flux at position x, time t. This quantity is defined as the number of vehicles passing through the location x, time t and has the unit of measure as the number of vehicles per unit length in a time unit.

We consider the problem of finding traffic density on a distance $\Delta x = x_2 - x_1$. Applying the flow conservation law, we have the equation [1]

$$\frac{\partial}{\partial}\int_{x_1}^{x_2} u(x,t)dx = J(x_1,t) - J(x_2,t), \quad (1)$$

This equation can be written as

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0 \quad (2)$$

where J is a function of traffic density and vehicle velocity, $J(x,t) = J(u, v(u))$, where v is the vehicle velocity, it depends on the traffic density, such as the Lighthill-Whitham-Richards model, which is established $v(u) = v_{max}\left(1 - \frac{u}{u_{max}}\right)$, or the Greenberg model, where the velocity is determined as $v(u) = a\log\frac{u_{max}}{u}$. where, u_{max}, v_{max} are the maximum limits of density and velocity over the distance considered, respectively.

Recently, there have been many works to find solutions for problem (2), in 2011, M. O. Gani [2] proposed the Lax-Friedrichs difference schema to find an approximate solution to equation (1) with the initial and Dirichlet boundary conditions. In 2013, Sultana, N., Parvin [3] gave a difference schema to find the solution to the problem, numerical results were given by the authors to illustrate the theory. In 2014, Torudonkumo [4] used the characteristic method to find the explicit solution of equation (1). Năm 2014. Recently, Thibault Liard [5] has found a solution to the problem for equation (1) in order to provide an optimal traffic control technical solution. In 2021, Simone Gottlich et al studied the problem model (2) when considering the time delay factor and gave the approximate finite difference schema for the problem and the properties of the approximate solution. Also in 2021, Dung N. D. [8] expands problem (2) by switching from homogeneous problem to heterogeneous problem, when the problem model is built in case of model error and experimental error, i.e. the right hand side of (2) will probably be a non-zero quantity and depend on the spatial position and temporal, so the problem is described by the equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = f(x,t) \quad (3)$$

Continuing to contribute to the published results, in this paper, we focus on finding a numerical solution to problem (3) when considering the time delay factor. Specifically, in Section 2, we build a regularization difference algorithm to find the solution to problem (3) in the condition of time delay. In section 3, we calculate the test to verify the correctness of the method given in section 2.

2. Method

In traffic density forecasting, delay time is important because traffic condition information takes some time to be transmitted from sensors, instrumentation or other data sources to the forecasting system. This can result from a variety of reasons such as the time it takes to collect data, the time it takes to process and transmit the data, as well as the time it takes for information to travel through the transport network. Thus, starting from previously published mathematical models, we consider the following specific problem [7]:

Given the numbers a, b, where a<b. The length of the road segment is $l = b - a$. Consider the problem in the domain

$Q_T = \{a < x < b; 0 < t \le T\}; \bar{Q}_T = \{a \le x \le b; 0 \le t \le T\}$.

The problem is to find the density function $u(x,t)$ satisfying

$$Lu \equiv \frac{\partial u(x,t)}{\partial t} + \frac{\partial J(x,t-\delta)}{\partial x} = f(x,t), (x,t) \in Q_T \tag{4}$$

$$u(x,0) = g(x), a < x < b \tag{5}$$

$$u(a,t) = g_a(t); u(b,t) = g_b(t), 0 < t \le T \tag{6}$$

Where, δ is the delay time, $g(x)$ is the traffic density at the beginning; $g_a(t)$ is the traffic density at $x = a$; $g_b(t)$ is the traffic density at $x = b$; $J(x,t) = J(u, v(u))$.

To find the solution to the problem (4)-(6), we approximate the differential problem by the difference problem on the difference grid. The difference grid is defined as follows:

Given the integers $N > 1, M \ge 1$ and set $h = \frac{b-a}{N}$, $x_i = a + ih$, $i = 0,1,2,...,N$, $\tau = \frac{T}{M}$, $t_j = j\tau$, $j = 0,1,2,...,M$. We divide the domain Q_T into cells by $x = x_i$ and $t = t_j$, Each point (x_i, t_j) is called a node and denoted by (i, j). h is called the spatial mesh step. τ is called the time grid step.

The goal of the method is to find the approximate solution of the problem at the nodes (i, j). The set of all nodes (i, j) forming a difference grid on $\bar{Q}_T$, for the calculation, we need to approximate the derivatives on the difference grid

$$\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} = \frac{\partial u}{\partial t}(x_i, t_j) + O(\tau), \tag{7}$$

$$\frac{J(x_{i+1}, t_j) - J(x_{i-1}, t_j)}{2h} = \frac{\partial J}{\partial x}(x_i, t_j) + O(h^2). \tag{8}$$

There are many ways of approximating the derivative, leading to many different ways to replace the differential problem by the difference problem. In this approximation, the results in [8] using the Lax-Friedrichs scheme approximate problem (4) by the difference problem

$$u_i^{j+1} = \frac{1}{2}\Big((u_{i-1}^j + u_{i+1}^j) - 2\gamma(J_{i+1}^j - J_{i-1}^j)\Big) + \tau f(x_i, t_j), \tag{9}$$

where $\gamma = \dfrac{\tau}{2h}$, $n_\delta = \dfrac{\delta}{\tau}$.

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In the case of a time delay, we replace (9) by the formula

$$u_i^{j+1} = \frac{1}{2}\Big((u_{i-1}^j + u_{i+1}^j) - 2\gamma(J_{i+1}^{j-n_\delta} - J_{i-1}^{j-n_\delta})\Big) + \tau f(x_i, t_j), \tag{10}$$

In order to increase the accuracy of each computational node, we regularize (10) by the regularization difference algorithm

$$u_i^{j+1} = \frac{1}{2}\Big(z_i^{j+1} - 2\gamma(J_{i+1}^{j-n_\delta} - J_{i-1}^{j-n_\delta})\Big) + \tau f(x_i, t_j),\ j = n_\delta, n_\delta + 1, \ldots \tag{11}$$

$$z_i^{j+1} = \alpha u_{i-1}^{j+1} + (1-\alpha)(u_{i-1}^j + u_{i+1}^j); 0 \le \alpha \le 1 \tag{12}$$

The stability and convergence of the algorithm are shown by the following theorem:

Theorem: *If* $sup\left\{\dfrac{\partial J}{\partial u} : x \in [a,b], t \in [0,\mathrm{T}]\right\} \le \dfrac{h}{\tau}$ *then algorithm (11) converges to the exact solution with degree of convergence is* $O(\tau + h^2)$

3. Experimental calculation

The velocity function is a density dependent function and has the form $v = v_{\max}(1 - u/u_{\max})$

Flux function: $J = uv(u)$.

Right-hand function: $f(x,t) = u_{max}\dfrac{x-b}{T(b-a)} + \dfrac{u_{max}v_{max}}{T(b-a)}\left(\dfrac{2t^2(x-b)}{T(b-a)} + t\right)$.

Density at the beginning: $u(x,0) = g(x) = u_{max}$

Density at the entrance: $u(a,t) = g_a(t) = u_{max}\left(1 - \dfrac{t}{T}\right)$

Density at exit: $u(b,t) = g_b(t) = u_{max}$.

With this data, we have the actual density $u = u_{max}\left(1 - \dfrac{t(b-x)}{T(b-a)}\right)$ and $v = v_{max}\left(1 - \dfrac{u}{u_{max}}\right)$, $J = uv(u)$, $\dfrac{\partial J}{\partial u} = v_{max}\left(1 - \dfrac{2u}{u_{max}}\right)$, $\sup\left(\dfrac{\partial J}{\partial u}\right) = v_{max}$.

In order to the iterative process to be stable and convergent, we divide the mesh to satisfy

$$\frac{\tau}{h} \le \frac{1}{\sup\left\{\dfrac{\partial J}{\partial u} : x \in [a,b], t \in [0,T]\right\}} = \frac{1}{v_{\max}}.$$

Given $a = 0$; $b = 10(km)$; $T = 10$ (hour); $u_{max} = 120$ (cars/km); $v_{max} = 80$ (km/hour); time delay $\delta = 7.2$ (seconds); Regularization parameter $\alpha = 10^{-4}$. The iterative process converges when $\frac{\tau}{h} \le \frac{1}{80}$. Divide the distance into N equal segments, the length of each segment is $h = \frac{b-a}{N} = 0.2\,(km)$. Table 1 is the calculation results to verify the convergence of the algorithm.

Table 1: *Calculation results of the algorithm (11)-(12)*

M	$\tau = T / M$ (hour)	$err \approx max\{u_i^j - u(x_i, t_j)\}$
10000	0.00100	108
20000	0.00050	95
50000	0.00020	64
100000	0.00010	33
1000000	0.00001	3

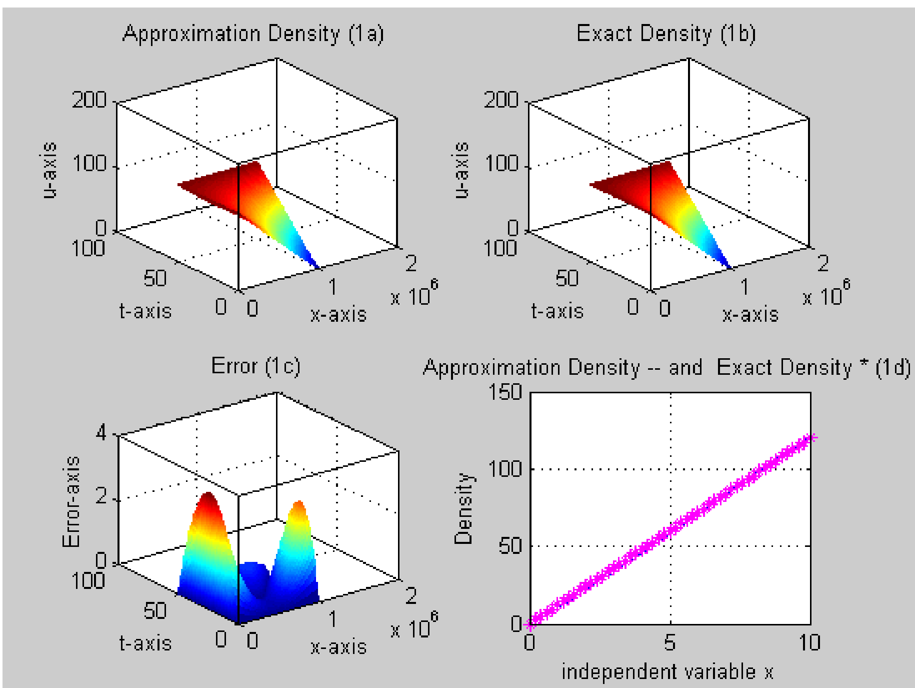

Figure 1: *Calculation results at N=50, M=1000,000*

The results in Table 1 show that when the M is arbitrarily large, the error between the approximate solution and the exact solution gradually decreases. In the above table, if the M=10.000, the error is 95 cars, when the number of time layer is gradually increased to 1000,000, the error is reduced to only 3 cars. Thus, this result shows that the proposed difference scheme is completely consistent with the proposed theory. The graph illustrating the calculation results by Figure 1, where figure d is the graph of the exact solution and the approximate solution at time T=10 (hours) also shows 2 lines describing the approximate solution and the exact solution close to each other, this proves the convergence of the method.

4. Conclusion

In this paper, we propose a regularization differential schema to find traffic density for the delay heterogeneity problem. This schema ensures stability and convergence with an accuracy of first order for the time grid step and second order for the spatial mesh step when setting the limiting condition on the ratio between these grid steps. The calculation results according to the schema have confirmed the convergence of the method and are consistent with the theory given in the paper. In the coming time, we will continue to study the method of choosing the regularization parameter to find the optimal parameter. Hopefully with these results, in the future we will continue to conduct theoretical research and test on a computational system that uses sensors to assist in data collection for a specific route.

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TÓM TẮT

THUẬT TOÁN SAI PHÂN HIỆU CHỈNH LAX FRIEDRICHS CHO BÀI TOÁN DỰ BÁO MẬT ĐỘ GIAO THÔNG CÓ TRỄ

Nguyễn Đình Dũng
Trường Đại học Công nghệ Thông tin và Truyền thông, Đại học Thái Nguyên, Việt Nam
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Trong những năm gần đây, dự báo mật độ giao thông đã đóng một vai trò quan trọng trong việc phát triển và cải thiện hiệu suất của hệ thống giao thông thông minh. Dự báo mật độ giao thông nhằm tối ưu hóa quản lý giao thông, quản lý đô thị giao thông phương tiện có khả năng điều phối giao thông, tối ưu hóa tín hiệu đèn giao thông và áp dụng quy định thông minh dựa trên dự báo, từ đó cải thiện khả năng xử lý đoạn đường và giảm thời gian di chuyển. Vì vậy, việc xây dựng các thuật toán dự đoán cùng với việc tích hợp vào hệ thống quản lý giao thông là cần thiết để thúc đẩy sự phát triển bền vững của hệ thống giao thông thông minh trong tương lai. Trong bài báo này, chúng tôi phát triển thuật toán dự đoán mật độ lưu lượng trên mô hình trễ Lighthill-Whitham-Richards, trong đó chúng tôi đề xuất phương pháp sai phân hiệu chỉnh làm cơ sở cho thuật toán. Chúng tôi chủ yếu tập trung xây dựng thuật toán và thực hiện cài đặt thử nghiệm để kiểm chứng tính đúng đắn của thuật toán trên mô hình toán học đã công bố.

Từ khóa: Mô hình luồng giao thông; sai phân Lax - Fedrichs; mô hình trễ LWR; sai phân hiệu chỉnh; mật độ giao thông.