# A COMMON FIXED POINT THEOREM FOR MAPPINGS SATISFYING *(E.A)* - PROPERTY VIA *C*-CLASS FUNCTIONS IN *b*-METRIC SPACES

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**ABSTRACT**

In this paper, we give and prove a common fixed point theorem for mappings satisfying (*E.A*)-property via *C*-class functions and some its corollaries in *b*-metric spaces. Our results are generalizations of some given results. Also, an example is given to illustrate the obtained main result.

**Keywords:** Common fixed point; *C*-class function; (*E.A*)-property; *b*-metric.

## 1. Introduction

The Banach contraction principle has been extended in many various directions, and many interesting fixed point results and coincidence point results were proposed. In 2014, Ansari *et al.* [1] used the *C*-class functions to generalize the Banach contraction principle in metric spaces and presented some the fixed point theorems in metric spaces. In 1989, the notion of *b*-metric spaces was introduced as a generalization of metric spaces [2]. After that, many extensions of the Banach contraction principle were studied in *b*-metric spaces and significant results including coincidence point results were obtained. In 2015, Huang *et al.* [3] proposed some coincidence point theorems for four maps in *b*-metric spaces. Recently, Radenovic *et al.* [10] used *T*-contractions in *b*-metric spaces to generalize the main results in [3].

In this paper, we use *C*-class functions [1] to generalize *T*-contractions [10], contractive conditions of rational type [4] and establish a coincidence point and common fixed point result in complete *b*-metric spaces. Our main result is a generalization of known ones in the literature. We also give an example to illustrate the obtained result.

First we recall some notions and properties that will be needed throughout the paper.

Definition 1.1 ([2]). Let $X$ be a nonempty set, let $s \geq 1$ be a real number, and let $d : X \times X \to [0,\infty)$ be a function such that for all $x, y, z \in X$,

1. $d(x, y) = 0$ if only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then $d$ is called a $b$-*metric* on $X$ and $(X, d, s)$ is called a $b$-*metric space*.

Obviously, a metric space is a $b$-metric space.

The following notions are introduced in [3], [5] and [10].

**Definition 1.2.** Let $(X, d, s)$ be a $b$-metric space with $s \geq 1$ and $f, g : X \longrightarrow X$ be mappings. Then

1. The pair $(f, g)$ is called *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, where $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

2. The pair $(f, g)$ is called *weakly compatible* if $fgx = gfx$ where $gx = fx$.

3. The pair $(f, g)$ is said to satisfy $(E.A)$-*property* if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

4. If $w = fx = gx$ for some $x$ in $X$, then $x$ is called a *coincidence point* of the pair $(f, g)$, where $w$ is called a *point of coincidence* of the pair $(f, g)$.

**Definition 1.3.** ([9]) Denote $\Psi$ the family of all *altering distance functions* $\psi : [0, \infty) \to [0, \infty)$, which satisfy the following

$(\psi_1)$ $\psi$ is non-decreasing and continuous;

$(\psi_2)$ $\psi(t) = 0$ if and only if $t = 0$.

**Definition 1.4.** ([6]) 1) Denote $\Phi$ the family of all *ultra altering distance functions* $\varphi : [0, \infty) \to [0, \infty)$, which satisfy the following

$(\varphi_1)$ $\varphi$ is non-decreasing and continuous;

$(\varphi_2)$ $\varphi(t) > 0$, if $t > 0$.

2) Denote $\Omega$ the family of all continuous functions $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$.

**Definition 1.5.** ([1]) A mapping $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a $C$-*class function* if it is continuous and satisfies following conditions

(i) $F(r, t) \leq r$; for all $r, t \in [0, \infty)$;

(ii) If $F(r, t) = r$, then either $r = 0$, or $t = 0$.

Denote $\mathcal{C}$ the family of all $C$-class functions. Note that for a given $F \in \mathcal{C}$, we have $F(0, 0) = 0$.

**Definition 1.6.** ([7]) Let $(X, d, s)$ be a $b$-metric space with $s \geq 1$. If $\{y_n\}$ is a convergent sequence whenever $\{Ty_n\}$ is convergent, then $T : X \to X$ is called *sequentially convergent*.

# 2 Main results

Now, we establish the following results.

**Theorem 2.1.** *Let $(X, d, s)$ be a b-metric space with $s \geq 1$ and $T, f, g, R, S : X \longrightarrow X$ be five mappings on $X$ such that the following conditions hold*

(1) *$T$ is an injective, continuous, and sequentially convergent mapping;*

(2) *There exist $\psi \in \Psi$, $\varphi \in \Phi$, $\omega \in \Omega$ and a $C$-class function $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that for all $x, y \in X$, we have*

$$\psi\left(sd\left(Tfx, Tgy\right)\right) \leq F\big(\psi\big(M_s^T(x, y)\big), \varphi\big(M_s^T(x, y)\big)\big) + \omega\big(\psi\big(L_s^T(x, y)\big)\big), \tag{1}$$

*where*

$$M_s^T(x, y) = \max\Big\{d(TSx, TRy), d(TSx, Tfx), d(TRy, Tgy), \frac{d(TSx, Tgy) + d(TRy, Tfx)}{2s}\Big\},$$
$$L_s^T(x, y) = \min\Big\{d(TSx, Tgy), d(TRy, Tfx), d(TSx, Tfx), d(TRy, Tgy)\Big\};$$

(3) *The pair $(f, S)$ or $(g, R)$ satisfies the $(E.A)$-property;*

(4) *$f(X) \subset R(X)$ and $g(X) \subset S(X)$;*

(5) *The range of one of the maps $f, g, R$ or $S$ is a closed subspace of $X$.*

*Then, the pairs $(f, S)$ and $(g, R)$ have a point of coincidence in $X$. Moreover, if*

(6) *$(f, S)$ and $(g, R)$ are weakly compatible,*

*then $f, g, R$ and $S$ have a unique common fixed point.*

*Proof.* 1) Firstly, we assume that the pair $(f, S)$ satisfies the $(E.A)$-property. Then, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = q$, for some $q \in X$. By assumption $f(X) \subset R(X)$, there is $\{y_n\} \subset X$ such that $fx_n = Ry_n$ for all $n \geq 1$. Hence, we have $\lim_{n\to\infty} Ry_n = q$. Now we shall prove that $\lim_{n\to\infty} Tgy_n = Tq$. Indeed, since $\psi$ is non-decreasing, by the contractive condition (1), and the property of $F$, we get

$$\begin{aligned}\psi\left(d\left(Tfx_n, Tgy_n\right)\right) &\leq \psi\left(sd\left(Tfx_n, Tgy_n\right)\right)\\ &\leq F\left(\psi\left(M_s^T\left(x_n, y_n\right)\right), \varphi\left(M_s^T\left(x_n, y_n\right)\right)\right) + \omega\big(\psi\big(L_s^T(x_n, y_n)\big)\big),\\ &\leq \psi\left(M_s^T\left(x_n, y_n\right)\right) + \omega\big(\psi\big(L_s^T(x_n, y_n)\big)\big),\end{aligned} \tag{2}$$

where

$$
\begin{aligned}
M_s^T(x_n,y_n) &= \max\Big\{d(TSx_n,TRy_n), d(TSx_n,Tfx_n), d(TRy_n,Tgy_n),\\
&\qquad \frac{d(TSx_n,Tgy_n)+d(TRy_n,Tfx_n)}{2s}\Big\}\\
&= \max\Big\{d(TSx_n,Tfx_n), d(TSx_n,Tfx_n), d(Tfx_n,Tgy_n),\\
&\qquad \frac{d(TSx_n,Tgy_n)+d(Tfx_n,Tfx_n)}{2s}\Big\} \qquad (3)\\
&\le \max\{d(TSx_n,Tfx_n), d(Tfx_n,Tgy_n),\\
&\qquad \frac{d(TSx_n,Tfx_n)+d(Tfx_n,Tgy_n)}{2}\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
L_s^T(x_n,y_n) &= \min\Big\{d(TSx_n,Tgy_n), d(Tfx_n,TRy_n), d(TSx_n,Tfx_n), d(TRy_n,Tgy_n)\Big\}\\
&= 0. \qquad (4)
\end{aligned}
$$

It follows from (2), (3) and (4) that

$$\liminf_{n\to\infty} d(Tfx_n,Tgy_n) = \liminf_{n\to\infty} M_s^T(x_n,y_n).$$

Without loss of generality, we assume that

$$\lim_{n\to\infty} d(Tfx_n,Tgy_n) = \lim_{n\to\infty} M_s^T(x_n,y_n).$$

By the properties of $\psi,\varphi,\omega$ and $F$, letting in (2) as $n\to\infty$ we obtain

$$
\begin{aligned}
\psi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)) &\le F(\psi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)), \varphi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)))\\
&\qquad + \omega\big(\psi\big(\lim_{n\to\infty} L_s^T(x_n,y_n)\big)\big)\\
&\le \psi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)).
\end{aligned}
$$

Thus, we have

$$F(\psi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)), \varphi(\lim_{n\to\infty} d(Tfx_n,Tgy_n))) = \psi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)).$$

By the properties of a $C$-class function $F$, this implies that $\psi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)) = 0$ or $\varphi(\lim_{n\to\infty} d(Tfx_n,Tgy_n)) = 0$. It follows from the properties of $\psi$ and $\varphi$ that $\lim_{n\to\infty} d(Tfx_n,Tgy_n) = 0$.

On the other hand, by assumption we have $\lim_{n\to\infty} Tfx_n = Tq$, Thus, from the inequality

$$0 \le d(Tq,Tgy_n) \le s[d(Tq,Tfx_n) + d(Tfx_n,Tgy_n)],$$

we have $\lim_{n\to\infty} Tgy_n = Tq$. Since $T$ is an injective, continuous, and sequentially convergent mapping, it follows that $\lim_{n\to\infty} gy_n = q$.

a) Now we assume that $R(X)$ is a closed subspace of $X$. Then, as $\lim_{n\to\infty} fx_n = q$ and $f(X) \subset R(X)$, there exists $r \in X$, such that $Rr = q$. By the contractive condition (1) and the properties of $F$, we get

$$\begin{aligned}\psi\left(sd\left(Tfx_n, Tgr\right)\right) &\le F\left(\psi\left(M_s^T\left(x_n, r\right)\right), \varphi\left(M_s^T\left(x_n, r\right)\right)\right) + \omega\big(\psi\big(L_s^T(x_n, r)\big)\big)\\ &\le \psi(M_s^T(x_n, r)) + \omega\big(\psi\big(L_s^T(x_n, r)\big)\big),\end{aligned} \tag{5}$$

where

$$\begin{aligned}M_s^T\left(x_n, r\right) &= \max\Big\{d\left(TSx_n, TRr\right), d\left(TSx_n, Tfx_n\right), d\left(TRr, Tgr\right),\\ &\qquad \frac{d\left(TSx_n, Tgr\right) + d\left(TRr, Tfx_n\right)}{2s}\Big\}\\ &= \max\Big\{d\left(TSx_n, Tq\right), d\left(TSx_n, Tfx_n\right), d\left(Tq, Tgr\right),\\ &\qquad \frac{d\left(TSx_n, Tgr\right) + d\left(Tq, Tfx_n\right)}{2s}\Big\}\\ &\le \max\Big\{d(TSx_n, Tq), s[d(TSx_n, Tq) + d(Tq, Tfx_n)], d(Tq, Tgr), \qquad (6)\\ &\qquad \frac{d(TSx_n, Tq) + d(Tq, Tgr)}{2} + \frac{d(Tq, Tfx_n)}{2s}\Big\}\end{aligned}$$

and

$$\begin{aligned}L_s^T(x_n, r) &= \min\Big\{d(TSx_n, Tgr), d(TRr, Tfx_n), d(TSx_n, Tfx_n), d(TRr, Tgr)\Big\}\\ &\le \min\Big\{d(TSx_n, Tgr), d(TRr, Tfx_n), s[d(TSx_n, Tq) + d(Tq, Tfx_n)], \qquad (7)\\ &\qquad d(TRr, Tgr)\Big\}.\end{aligned}$$

It follows from (6) and (7) that $\limsup_{n\to\infty} M_s^T(x_n, r) = d(Tq, Tgr)$ and $\lim_{n\to\infty} L_s^T(x_n, r) = 0$. Hence, by continuity of $\psi, \varphi, \omega$ and $F$, taking the upper limit in (5) as $n \to \infty$, we obtain

$$\begin{aligned}\psi(\limsup_{n\to\infty} sd(Tfx_n, Tgr) &\le F(\psi(\limsup_{n\to\infty} M_s^T(x_n, r)), \varphi(\limsup_{n\to\infty} M_s^T(x_n, r)))\\ &\qquad + \omega\big(\psi\big(\limsup_{n\to\infty} L_s^T(x_n, r)\big)\big)\\ &= F(\psi(d(Tq, Tgr)), \varphi(d(Tq, Tgr))) \le \psi(d(Tq, Tgr)).\end{aligned} \tag{8}$$

On the other hand, we have $d(Tq, Tgr) \le s[d(Tq, Tfx_n) + d(Tfx_n, Tgr)]$. Thus, we get

$$d(Tq, Tgr) \le \limsup_{n\to\infty} sd(Tfx_n, Tgr).$$

Since $\psi$ is non-decreasing, we have

$$\psi(d(Tq,Tgr)) \leq \psi(\limsup_{n\to\infty} sd(Tfx_n,Tgr)). \tag{9}$$

Therefore, from (8) and (9) we obtain that

$$F(\psi(d(Tq,Tgr)),\varphi(d(Tq,Tgr))) = \psi(d(Tq,Tgr)).$$

By the properties of a $C$-class function $F$, it implies that $\psi(d(Tq,Tgr)) = 0$ or $\varphi(d(Tq,Tgr)) = 0$, i.e $Tq = Tgr$. Since $T$ is injective, this implies that $q = gr$. Thus, $r$ is a coincidence point of the pair of mappings $(g,R)$.

Similarly, since $\lim\limits_{n\to\infty} gy_n = q$, $gr = q$ and $g(X) \subset S(X)$, there exists $z \in X$ such that $q = Sz$. We will show that $Sz = fz$. Indeed, by the contractive condition (1), the properties of $F$, and $\psi$ is non-decreasing, we have

$$\begin{aligned}\psi\left(d\left(Tfz,Tq\right)\right) &\leq \psi\left(sd\left(Tfz,Tq\right)\right) = \psi\left(sd\left(Tfz,Tgr\right)\right)\\ &\leq F\left(\psi\left(M_s^T\left(z,r\right)\right),\varphi\left(M_s^T\left(z,r\right)\right)\right) + \omega\big(\psi\big(L_s^T(z,r)\big)\big)\\ &\leq \psi\left(M_s^T(z,r)\right) + \omega\big(\psi\big(L_s^T(z,r)\big)\big),\end{aligned} \tag{10}$$

where

$$\begin{aligned}M_s^T\left(z,r\right) &= \max\left\{d\left(TSz,TRr\right), d\left(TSz,Tfz\right), d\left(TRr,Tgr\right), \frac{d\left(TSz,Tgr\right)+d\left(TRr,Tfz\right)}{2s}\right\}\\ &= \max\left\{d\left(Tq,Tq\right), d\left(Tfz,Tq\right), d\left(Tq,Tq\right), \frac{d\left(Tq,Tq\right)+d\left(Tfz,Tq\right)}{2s}\right\}\\ &= \max\left\{d\left(Tfz,Tq\right), \frac{d\left(Tfz,Tq\right)}{2s}\right\} = d\left(Tfz,Tq\right),\end{aligned} \tag{11}$$

and

$$\begin{aligned}L_s^T(z,r) &= \min\Big\{d(TSz,Tgr), d(TRr,Tfz), d(TSz,Tfz), d(TRr,Tgr)\Big\}\\ &= \min\Big\{d(Tq,Tq), d(Tq,Tfz), d(Tq,Tfz), d(Tq,Tq)\Big\} = 0.\end{aligned} \tag{12}$$

Hence, by (10) - (12) and the properties of $\omega$, we get

$$\begin{aligned}\psi\left(d\left(Tfz,Tq\right)\right) &\leq \psi\left(sd\left(Tfz,Tq\right)\right) = \psi\left(sd\left(Tfz,Tgr\right)\right)\\ &\leq F\left(\psi\left(M_s^T\left(z,r\right)\right),\varphi\left(M_s^T\left(z,r\right)\right)\right) + \omega\big(\psi\big(L_s^T(z,r)\big)\big)\\ &\leq F\left(\psi\left(d(Tfz,Tq)\right),\varphi\left(d(Tfz,Tq)\right)\right) \leq \psi\left(d(Tfz,Tq)\right).\end{aligned}$$

This implies that $F\left(\psi\left(d\left(Tfz,Tq\right)\right),\varphi\left(d\left(Tfz,Tq\right)\right)\right) = \psi\left(d\left(Tfz,Tq\right)\right)$. It follows from the last inequality and the properties of a $C$-class function $F$ that $\psi\left(d\left(Tfz,Tq\right)\right) = 0$ or $\varphi\left(d\left(Tfz,Tq\right)\right) = 0$. By the properties of $\psi$ and $\varphi$, we get $d\left(Tfz,Tq\right) = 0$, i.e

$Tfz = Tq$. Since $T$ is injective, this implies that $fz = q$, and we have $Sz = fz = q$. Thus, $z$ is a coincidence point of the pair mappings $(f, S)$. Therefore, from the above arguments we obtain that $fz = Sz = gr = Rr = q$. This shows that $q$ is a point of coincidence of the pairs $(f, S)$ and $(g, R)$.

We now assume that the pairs $(f, S)$ and $(g, R)$ are weakly compatible. Then, by weak compatibility of $(f, S)$ and $(g, R)$, we get $fSz = Sfz$ and $gRr = Rgr$, this means that $fq = Sq$ and $gq = Rq$.

Next, we shall prove that $q$ is a common fixed point of $f, g, R$ and $S$. Indeed, by contractive condition (1), the properties of $F$, since $\psi$ is non-decreasing, we have

$$\begin{aligned}\psi\left(d\left(Tfq, Tq\right)\right) &= \psi\left(d\left(Tfq, Tgr\right)\right) \leq \psi\left(sd\left(Tfq, Tgr\right)\right)\\ &\leq F\left(\psi\left(M_s^T\left(q, r\right)\right), \varphi\left(M_s^T\left(q, r\right)\right)\right) + \omega\big(\psi\big(L_s^T(q, r)\big)\big)\\ &\leq \psi\left(M_s^T\left(q, r\right)\right) + \omega\big(\psi\big(L_s^T(q, r)\big)\big),\end{aligned} \tag{13}$$

where

$$\begin{aligned}M_s^T\left(q, r\right) &= \max\Big\{d\left(TSq, TRr\right), d\left(TSq, Tfq\right), d\left(TRr, Tgr\right),\\ &\qquad\qquad \frac{d\left(TSq, Tgr\right) + d\left(TRr, Tfq\right)}{2s}\Big\}\\ &= \max\left\{d\left(Tfq, Tq\right), d\left(Tfq, Tfq\right), d\left(Tq, Tq\right), \frac{d\left(Tfq, Tq\right) + d\left(Tq, Tfq\right)}{2s}\right\}\\ &= d\left(Tfq, Tq\right),\end{aligned} \tag{14}$$

and

$$L_s^T(q, r) = \min\Big\{d(TSq, Tgr), d(TRr, Tfq), d(TSq, Tfq), d(TRr, Tgr)\Big\} = 0. \tag{15}$$

Hence, by (13) - (15) and the properties of $\omega$, we get

$$\psi\left(d\left(Tfq, Tq\right)\right) \leq F\left(\psi\left(d\left(Tfq, Tq\right)\right), \varphi\left(d\left(Tfq, Tq\right)\right)\right) \leq \psi\left(d\left(Tfq, Tq\right)\right),$$

which implies that

$$F\left(\psi\left(d\left(Tfq, Tq\right)\right), \varphi\left(d\left(Tfq, Tq\right)\right)\right) = \psi\left(d\left(Tfq, Tq\right)\right).$$

From this last equality, by the properties of $C$-class function $F$, we obtain $\psi\left(d\left(Tfq, Tq\right)\right) = 0$ or $\varphi\left(d\left(Tfq, Tq\right)\right) = 0$. It follows from the properties of $\psi$ and $\varphi$ that $d\left(Tfq, Tq\right) = 0$. This means that $Tfq = Tq$. Since $T$ is injective, this implies that $fq = q$. Thus, we get $fq = Sq = q$.

Similarly, by contractive condition (1), the properties of $F$, since $\psi$ is non-decreasing, we have

$$\begin{aligned}\psi\left(d\left(Tq, Tgq\right)\right) &= \psi\left(d\left(Tfz, Tgq\right)\right) \leq \psi\left(sd\left(Tfz, Tgq\right)\right)\\ &\leq F\left(\psi\left(M_s^T\left(z, q\right)\right), \varphi\left(M_s^T\left(z, q\right)\right)\right) + \omega\big(\psi\big(L_s^T(z, q)\big)\big)\\ &\leq \psi\left(M_s^T(z, q)\right) + \omega\big(\psi\big(L_s^T(z, q)\big)\big),\end{aligned} \tag{16}$$

where

$$M_s^T(z,q) = \max\Big\{d(TSz,TRq), d(TSz,Tfz), d(TRq,Tgq), \frac{d(TSz,Tgq)+d(TRq,Tfz)}{2s}\Big\}$$
$$= \max\Big\{d(Tq,Tgq), 0, 0, \frac{d(Tq,Tgq)+d(Tq,Tgq)}{2s}\Big\} = d(Tq,Tgq), \quad (17)$$

and

$$L_s^T(z,q) = \min\Big\{d(TSz,Tgq), d(TRq,Tfz), d(TSz,Tfz), d(TRq,Tgq)\Big\} = 0. \quad (18)$$

Therefore, by (16) - (18) and the properties of $\omega$, we have

$$\psi(d(Tq,Tgq)) \leq F(\psi(d(Tq,Tgq)), \varphi(d(Tq,Tgq))) \leq \psi(d(Tq,Tgq)),$$

which implies that

$$F(\psi(d(Tq,Tgq)), \varphi(d(Tq,Tgq))) = \psi(d(Tq,Tgq)).$$

From this last equality, by the properties of $C$-class function $F$, we obtain $\psi(d(Tq,Tgq)) = 0$ or $\varphi(d(Tq,Tgq)) = 0$. It follows from the properties of $\psi$ and $\varphi$ that $d(Tq,Tgq) = 0$, i.e $Tq = Tgq$. Since $T$ is injective, this implies that $gq = q$. Thus, we get $Rq = gq = q$. Therefore $q$ is a common fixed point of $f, g, R, S$.

Finally, we prove that $q$ a unique common fixed point. Assume that $p$ is another common fixed point of $f, g, R, S$. Then, by contractive condition (1), the properties of $F$, since $\psi$ is non-decreasing, we have

$$\begin{aligned}\psi(d(Tq,Tp)) &= \psi(d(Tfq,Tgp)) \leq \psi(sd(Tfq,Tgp)) \\ &\leq F\big(\psi\big(M_s^T(q,p)\big), \varphi\big(M_s^T(q,p)\big)\big) + \omega\big(\psi\big(L_s^T(q,p)\big)\big) \\ &\leq \psi\big(M_s^T(q,p)\big) + \omega\big(\psi\big(L_s^T(q,p)\big)\big),\end{aligned}$$

where

$$M_s^T(q,p) = \max\Big\{d(TSq,TRp), d(TSq,Tfq), d(TRp,Tgp), \frac{d(TSq,Tgp)+d(TRp,Tfq)}{2s}\Big\}$$
$$= \max\Big\{d(Tq,Tp), d(Tq,Tq), d(Tp,Tp), \frac{d(Tq,Tp)+d(Tq,Tp)}{2s}\Big\}$$
$$= d(Tq,Tp),$$

and

$$L_s^T(q,p) = \min\Big\{d(TSq,Tgp), d(TRp,Tfq), d(TSq,Tfq), d(TRp,Tgp)\Big\} = 0. \quad (19)$$

Hence, we get

$$\psi\left(d\left(Tq,Tp\right)\right)\leq F\left(\psi\left(d\left(Tq,Tp\right)\right),\varphi\left(d\left(Tq,Tp\right)\right)\right)\leq\psi\left(d(Tq,Tp)\right),$$

which implies that

$$F\left(\psi\left(d\left(Tq,Tp\right)\right),\varphi\left(d\left(Tq,Tp\right)\right)\right)=\psi\left(d(Tq,Tp)\right).$$

Since $F$ is a $C$-class function, it follows that $\psi\left(d\left(Tq,Tp\right)\right)=0$ or $\varphi\left(d\left(Tq,Tp\right)\right)=0$. By the properties of $\psi$ and $\varphi$, it follows that $d\left(Tq,Tp\right)=0$, this means that $Tq=Tp$. Since $T$ is injective, this implies that $q=p$. Thus, $q$ is the unique common fixed point of $f,g,R,S$.

b) Assume that $f\left(X\right)$ is a closed subspace of $X$. Then, since $\lim\limits_{n\to\infty}fx_n=q$, we have $q\in f(X)$. Again as $f\left(X\right)\subset R\left(X\right)$, it follows that $q\in R(X)$. Hence, there exists $r\in X$ such that $Rr=q$. Using similar arguments in the case that $R(X)$ is a closed space of $X$, we conclude that $f,g,R,S$ have a unique common fixed point in $X$.

c) In the case that $S(X)$ is a closed subspace of $X$, we use arguments what is similar to one in the case a).

d) In the case that $g(X)$ is a closed subspace of $X$, its proof is similar to one in the case b).

2) Finally, in the case that the pair $(g,R)$ satisfies the $(E.A)$-property, its proof is similar to one when the pair $(f,S)$ satisfies the $(E.A)$-property. □

If we choose $g=f$ and $R=S$ in Theorem 2.1, then we obtain the following result.

**Corollary 2.2.** *Let $(X,d,s)$ be a b-metric space with $s\geq1$ and $T,f,S:X\longrightarrow X$ be three mappings on $X$ such that the following conditions hold*

(1) *$T$ is an injective, continuous, and sequentially convergent mapping;*

(2) *There exist $\psi\in\Psi$, $\varphi\in\Phi$, $\omega\in\Omega$ and a $C$-class function $F:[0,\infty)\times[0,\infty)\to\mathbb{R}$ such that for all $x,y\in X$, we have*

$$\psi\left(sd\left(Tfx,Tfy\right)\right)\leq F\Big(\psi\big(M_s^T(x,y)\big),\varphi\big(M_s^T(x,y)\big)\Big)+\omega\Big(\psi\big(L_s^T(x,y)\big)\Big),\quad(20)$$

*where*

$$M_s^T(x,y)=\max\Big\{d(TSx,TSy),d(TSx,Tfx),d(TSy,Tfy),\frac{d(TSx,Tfy)+d(TSy,Tfx)}{2s}\Big\},$$
$$L_s^T(x,y)=\min\Big\{d(TSx,Tfy),d(TSy,Tfx),d(TSx,Tfx),d(TSy,Tfy)\Big\};$$

(3) *The pair $(f,S)$ satisfies the $(E.A)$-property;*

(4) $f(X) \subset S(X)$;

(5) *The range of one in the maps $f, S$ is a closed subspace of $X$.*

*Then, the pair $(f, S)$ have a point of coincidence in $X$. Moreover, if*

(6) $(f, S)$ *is weakly compatible,*

*then $f, S$ have a unique common fixed point.*

In Theorem 2.1, by choosing $T : X \longrightarrow X$ defined by $Tx = x$ for all $x \in X$, the function $\omega(t) = 0$ for all $t \in [0, \infty)$, and the $C$-class function $F$ defined by $F(r, t) = r - t$ for all $r, t \in [0, \infty)$, we obtain the following result.

**Corollary 2.3.** (Theorem 2.1 [9]) *Let $(X, d, s)$ be a b-metric space and $f, g, R, S : X \to X$ be four self-mappings on $X$ satisfying $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ such that the following condition holds*

$$\psi\left(s^2 d\left(fx, gy\right)\right) \leq \psi\left(M_s\left(x, y\right)\right) - \varphi\left(M_s\left(x, y\right)\right), \tag{21}$$

*for all $x, y \in X$, where $\psi, \varphi \in \Psi$ and*

$$M_s(x, y) = \max\left\{d(Sx, Ry), d(fx, Sx), d(gy, Ry), \frac{d(fx, Ry) + d(Sx, gy)}{2s}\right\}.$$

*Suppose that one of the pairs $(f, S)$ and $(g, R)$ satisfies the $(E.A)$-property and one of subspaces $f(X), g(X), R(X)$ and $S(X)$ is closed in $X$. Then, the pairs $(f, S)$ and $(g, R)$ have a point of coincidence in $X$. Moreover, if the pairs $(f, S)$ and $(g, R)$ are weakly compatible, then $f, g, R, S$ have a unique common fixed point.*

In Theorem 2.1, by choosing $T : X \longrightarrow X$ defined by $Tx = x$ for all $x \in X$, and the functions $\psi \in \Psi$ and $\varphi \in \Phi$ defined by $\psi(t) = \varphi(t) = t$, $\omega(t) = 0$ for all $t \in [0, \infty)$, and the $C$-class function $F$ by $F(r, t) = s^{1-\varepsilon} r$ for all $r, t \in [0, \infty)$, for some $\varepsilon > 1$, we obtain the following result.

**Corollary 2.4.** (Theorem 10 [8]) *Let $(X, d, s)$ be a b-metric space with $s > 1$ and $f, g, R, S : X \to X$ be four self-mappings on $X$ satisfying $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ such that the following condition holds*

$$s^{\varepsilon} d\left(fx, gy\right) \leq M_s\left(x, y\right), \tag{22}$$

*for all $x, y \in X$, with $\varepsilon > 1$ and*

$$M_s(x, y) = \max\left\{d(Sx, Ry), d(fx, Sx), d(gy, Ry), \frac{d(fx, Ry) + d(Sx, gy)}{2s}\right\}.$$

*Suppose that one of the pairs $(f, S)$ and $(g, R)$ satisfies the $(E.A)$-property and one of subspaces $f(X), g(X), R(X)$ and $S(X)$ is closed in $X$. Then, the pairs $(f, S)$ and $(g, R)$ have a point of coincidence in $X$. Moreover, if the pairs $(f, S)$ and $(g, R)$ are weakly compatible, then $f, g, R, S$ have a unique common fixed point.*

**Example 2.5.** Let $X = [0,1]$. Consider

1. the function $d : X \times X \to [0,\infty)$ defined by
$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$
2. the self-maps $T, f, g, R, S : X \to X$ defined by
$$T(x) = \frac{x}{2}, \qquad f(x) = \frac{x}{4}, \qquad g(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2}; \\ \frac{1}{8} & \text{if } x = \frac{1}{2}, \end{cases}$$
and
$$S(x) = \begin{cases} 2x & \text{if } \ 0 \leq x \leq \frac{1}{2}; \\ \frac{1}{8} & \text{if } \ \frac{1}{2} < x \leq 1, \end{cases} \qquad R(x) = \begin{cases} x & \text{if } \ 0 \leq x < \frac{1}{2}; \\ \frac{1}{2} & \text{if } \ \frac{1}{2} \leq x \leq 1, \end{cases}$$
for all $x \in X$.

Then we have

(a) $(X, d)$ is a $b$-metric space with $s = 2$.

(b) There exist $\omega \in \Omega$, $\psi \in \Psi$, $\varphi \in \Phi$, and $F \in \mathcal{C}$ such that Theorem 2.1 can apply to $T, f, g, S, R$.

*Chứng minh.* (a) $(X, d)$ is a $b$-metric space with $s = 2$, see [8].

(b) It is easy to see that $T$ is an injective, continuous, and sequentially convergent mapping, $f(X)$ is closed and $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$. By choosing $\{x_n\} \subset X$, with $x_n = \frac{1}{2} + \frac{1}{n}$ for all $n \geq 1$, we have $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Sx_n = \frac{1}{8}$. Also, the pair of maps $(f, S)$ satisfies the $(E.A)$-property, but they are not compatible, because $\lim\limits_{n\to\infty} d(fSx_n, Sfx_n) \neq 0$.

We now choose a function $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by $F(s,t) = \frac{99}{100}s$ for all $s, t \in [0,\infty)$, the altering distance functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = \sqrt{t}$, $\varphi(t) = t$ for all $t \in [0,\infty)$, and the function $\omega(t) = 0$ for all $t \in [0,\infty)$. Then, it is easy to check that $F$ is a $C$-class function and the contractive condition (1) is satisfied for all $x, y \in X$. Indeed,

(i) If $x = 0, y = \frac{1}{2}$, then $Tfx = 0,\ Tgy = \frac{1}{16}, TSx = 0,\ TRy = \frac{1}{4}$. Then, we have $d(Tfx, Tgy) = \left(\frac{1}{16}\right)^2$ and $d(TSx, TRy) = \left(\frac{1}{4}\right)^2$. Hence
$$\begin{aligned}\psi\big(2d(Tfx,Tgy)\big) = \frac{\sqrt{2}}{16} \leq \frac{99}{100}.\frac{1}{4} = \frac{99}{100}\psi\big(d(TSx,TRy)\big) \leq \frac{99}{100}\psi\big(M_s^T(x,y)\big) \\ \leq F\big(\psi\big(M_s^T(x,y)\big), \varphi\big(M_s^T(x,y)\big)\big).\end{aligned}$$

(ii) If $x = 0, y \neq \frac{1}{2}$, then $Tfx = Tgy = 0$. Then, we get $d(Tfx, Tgy) = 0$. Hence, we have

$$\psi\big(2d(Tfx, Tgy)\big) = 0 \leq F\big(\psi\big(M_s^T(x,y)\big), \varphi\big(M_s^T(x,y)\big).$$

(iii) If $x = y = \frac{1}{2}$, then $Tfx = Tgy = \frac{1}{16}, TSx = \frac{1}{2}, TRy = \frac{1}{4}$ and we get $d(Tfx, Tgy) = \left(\frac{2}{16}\right)^2 = \left(\frac{1}{8}\right)^2$ and $d(TSx, TRy) = \left(\frac{1}{2} + \frac{1}{4}\right)^2 = \left(\frac{3}{4}\right)^2$. Hence, we have

$$\begin{aligned}\psi\big(2d(Tfx, Tgy)\big) = \frac{\sqrt{2}}{8} \leq \frac{99}{100}.\frac{3}{4} = \frac{99}{100}\psi\big(d(TSx, TRy)\big) \leq \frac{99}{100}\psi\big(M_s^T(x,y)\big) \\ \leq F\big(\psi\big(M_s^T(x,y)\big), \varphi\big(M_s^T(x,y)\big)\big).\end{aligned}$$

(iv) If $x = \frac{1}{2}, y \neq \frac{1}{2}$, then $Tfx = \frac{1}{16},\ Tgy = 0,\ TSx = \frac{1}{2}$. Then $d(Tfx, Tgy) = \left(\frac{1}{16}\right)^2$, and $d(TSx, Tfx) = \left(\frac{1}{2} + \frac{1}{16}\right)^2 = \left(\frac{9}{16}\right)^2$. Hence, we obtain

$$\begin{aligned}\psi\big(2d(Tfx, Tgy)\big) = \frac{\sqrt{2}}{16} \leq \frac{99}{100}.\frac{9}{16} = \frac{99}{100}\psi\big(d(TSx, Tfx)\big) \leq \frac{99}{100}\psi\big(M_s^T(x,y)\big) \\ \leq F\big(\psi\big(M_s^T(x,y)\big), \varphi\big(M_s^T(x,y)\big)\big).\end{aligned}$$

(v) If $x \in \left(0, \frac{1}{2}\right), y = \frac{1}{2}$, then $Tfx = \frac{x}{8},\ Tgy = \frac{1}{16},\ TSx = x,\ TRy = \frac{1}{4}$. Then, we have $d(Tfx, Tgy) = \left(\frac{x}{8} + \frac{1}{16}\right)^2$ and $d(TSx, TRy) = \left(x + \frac{1}{4}\right)^2$. Therefore, we get

$$\begin{aligned}\psi\big(2d(Tfx, Tgy)\big) = \sqrt{2}\left(\frac{x}{8} + \frac{1}{16}\right) \leq \frac{99}{100}.\left(x + \frac{1}{4}\right) = \frac{99}{100}\psi\big(d(TSx, TRy)\big) \\ \leq \frac{99}{100}\psi\big(M_s^T(x,y)\big) \\ \leq F\big(\psi\big(M_s^T(x,y)\big), \varphi\big(M_s^T(x,y)\big)\big).\end{aligned}$$

(vi) If $x \in \left(0, \frac{1}{2}\right), y \neq \frac{1}{2}$, then $Tfx = \frac{x}{8},\ Tgy = 0,\ TSx = x$. Then, we have $d(Tfx, Tgy) = \left(\frac{x}{8}\right)^2$ and $d(TSx, Tfx) = \left(x + \frac{x}{8}\right)^2 = \left(\frac{9x}{8}\right)^2$. Hence, we get

$$\begin{aligned}\psi\big(2d(fx, gy)\big) = \frac{\sqrt{2}x}{8} \leq \frac{99}{100}.\frac{9x}{8} = \frac{99}{100}\psi\big(d(TSx, Tfx)\big) \\ \leq \frac{99}{100}\psi\big(M_s^T(x,y)\big) = F\big(\psi\big(M_s^T(x,y)\big), \varphi\big(M_s^T(x,y)\big)\big).\end{aligned}$$

(vii) If $x \in \left(\frac{1}{2}, 1\right]$, $y = \frac{1}{2}$, then $Tfx = \frac{x}{8}$, $Tgy = \frac{1}{16}$, $TRy = \frac{1}{4}$. Then $d(Tfx, Tgy) = \left(\frac{x}{8} + \frac{1}{16}\right)^2$, and $d(TRy, Tgy) = \left(\frac{1}{4} + \frac{1}{16}\right)^2$. Also, we have

$$\begin{aligned}\psi\big(2d(fx, gy)\big) = \sqrt{2}\left(\frac{x}{8} + \frac{1}{16}\right) &\leq \frac{99}{100}.\left(\frac{1}{4} + \frac{1}{16}\right) = \frac{99}{100}\psi\big(d(TRy, Tgy)\big)\\ &\leq \frac{99}{100}\psi\big(M_s^T(x, y)\big)\\ &\leq F\big(\psi\big(M_s^T(x, y)\big), \varphi\big(M_s^T(x, y)\big)\big).\end{aligned}$$

(viii) If $x \in \left(\frac{1}{2}, 1\right]$, $y \neq \frac{1}{2}$, then $Tfx = \frac{x}{8}$, $Tgy = 0$, $TSx = \frac{1}{16}$. Then, we have $d(Tfx, Tgy) = \left(\frac{x}{8}\right)^2$, and $d(TSx, Tfx) = \left(\frac{x}{8} + \frac{1}{16}\right)^2$. Hence, we have

$$\begin{aligned}\psi\big(2d(fx, gy)\big) = \frac{\sqrt{2}x}{8} &\leq \frac{99}{100}.\left(\frac{x}{8} + \frac{1}{16}\right) = \frac{99}{100}\psi\big(d(TSx, Tfx)\big)\\ &\leq \frac{99}{100}\psi\big(M_s^T(x, y)\big) = F\big(\psi\big(M_s^T(x, y)\big), \varphi\big(M_s^T(x, y)\big)\big).\end{aligned}$$

It is to see that the pairs of maps $(f, S)$ and $(g, R)$ are weakly compatible. Hence, we see that all conditions of Theorem 2.1 are satisfied for all $x, y \in X$. Therefore, $f, g, R, S$ have a unique common fixed point. Moreover, $0$ is the unique common fixed point of the maps $f, g, R$ and $S$.

□

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## TÓM TẮT

## MỘT ĐỊNH LÝ ĐIỂM BẤT ĐỘNG CHUNG CỦA CÁC ÁNH XẠ THỎA MÃN TÍNH CHẤT-$(E.A)$ NHỜ CÁC HÀM $C$-LỚP TRONG CÁC KHÔNG GIAN $b$-MÊTRIC

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Trong bài báo này, chúng tôi đưa ra và chứng minh một định lý điểm bất động chung cho các ánh xạ thỏa mãn tính chất-$(E.A)$ nhờ các hàm $C$-lớp và một vài hệ quả của nó trong các không gian $b$-mêtric. Kết quả của chúng tôi là mở rộng của một số kết quả đã có. Chúng tôi cũng đưa ra một ví dụ để minh họa kết quả chính đã thu được.

**Từ khóa.** Điểm bất động chung; hàm $C$-lớp; tính chất-$(E.A)$; $b$-mêtric.