

FINITE CONVERGENCE OF THE PROXIMAL POINT METHOD FOR SOLVING EQUILIBRIUM PROBLEMS ON HADAMARD MANIFOLDS

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In this paper, we prove the finite convergence of sequences generated by (inexact and exact) proximal point methods for solving pseudomonotone equilibrium problems on Hadamard manifolds under the linear conditioning of the solution set.

Keywords: Equilibrium problems; Hadamard manifolds; pseudomonotone bifunction; linear conditioning; finite convergence; proximal point method.

1. Introduction

The equilibrium problem, also known as Ky Fan inequality, introduced by E. Blum and W. Oettli ([1]), is a generalization of several problems such as optimization problems, variational inequality problems, complementary problems, saddle point problems, fixed point problems, Nash equilibrium problem in noncooperative games, ... (see, e.g., in [1], [2] and references therein). Because of its applications in economics, transportations, networks, image reconstructions, elasticity, etc. EP has been studied extensively by many authors. Two basic and important issues for EP are the existence of solutions and iterative methods for finding solutions. There have been a large number of papers dealing with above-mentioned issues in the literature (see, e.g., [3], [1], [4], [5], [6], [7] and references therein).

In 2012, V. Colao et al. ([8]) were first introduced the equilibrium problem on manifolds. In that paper, the authors proved the existence of solutions for equilibrium problems on Hadamard manifolds. They also proved the convergence of Picard iteration for firmly nonexpansive mappings in the setting of Hadamard manifolds and used to devise an algorithm to approximate solutions. The results in [8] were then improved and extended to general Riemannian manifolds by Wang et al. ([9]) with new approach. For other existence results for solutions of equilibrium problems, we refer the reader to, e.g., [10], [11], [12] and references therein. Some methods for solving equilibrium problems on manifolds can be found in [10], [13], [14], [15], [16]. Above-mentioned results are extensions of analogous results from linear spaces to

manifolds. Note that manifolds do not have linear structure and many known properties and techniques used in linear spaces do not work in the setting of manifolds. So new techniques are needed when we deal with equilibrium problems on manifolds.

A. Moudafi [17] introduced the concept of θ -conditioning for equilibrium problems. This concept extends and unifies several concepts and conditions in optimization theory such as weak sharp solutions or weak sharp minima (see, e.g., [18], [19], [20], [21]). In [17], the author proved the finite convergence of some iterative methods solving equilibrium problems in Hilbert spaces under the assumption that the considered equilibrium bifunction satisfies the θ -conditioning. Recently, the authors in [6] provided some characterizations for the linear conditioning (1-conditioning) and studied the finite convergence of some methods solving equilibrium problems. In that paper, they also gave an upper bound for the number of iterations in which the sequence generated by the proximal point method converges to a solution of the considered equilibrium problem.

The aim of this paper is to investigate the finite convergence of the proximal point method for solving equilibrium problems on Hadamard manifolds under the linear conditioning assumption of the solution set. Our result extends the result in [6] from linear spaces to Hadamard manifolds.

2 Preliminaries

In this section, we introduce some basic definitions, notions and results about Riemannian geometry which can be found in, for instances, [22], [23], [24].

Let \mathcal{M} be a connected finite-dimensional smooth manifold. We denote by $\mathcal{T}_x\mathcal{M}$ the tangent space of \mathcal{M} at a point $x \in \mathcal{M}$ and by $\mathcal{TM} = \bigcup_{x \in \mathcal{M}} \mathcal{T}_x\mathcal{M}$ the tangent bundle of \mathcal{M} . Note that $\mathcal{T}_x\mathcal{M}$ is a vector space with the same dimension as \mathcal{M} and \mathcal{TM} is naturally a manifold. We suppose that \mathcal{M} is endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle \cdot, \cdot \rangle_x$ the inner product on $\mathcal{T}_x\mathcal{M}$ and by $\| \cdot \|_x$ the corresponding norm to the inner product $\langle \cdot, \cdot \rangle_x$. If no confusion occurs, the subscript x is omitted.

The length of a piecewise smooth curve $\gamma : [a, b] \rightarrow \mathcal{M}$ joining x to y in \mathcal{M} , i.e., $x = \gamma(a)$ and $y = \gamma(b)$, is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\| dt,$$

where $\gamma'(t) \in \mathcal{T}_{\gamma(t)}\mathcal{M}$ is a tangent vector. The Riemannian distance $d(x, y)$ between x and y is the minimal length of all such curves connecting x and y . This distance induces the original topology on \mathcal{M} .

Let ∇ be the Levi-Civita connection associated with the Riemannian metric. A vector field V is said to be parallel along a smooth curve γ if $\nabla_{\gamma'(t)}V = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' is parallel along γ , i.e., $\nabla_{\gamma'(t)}\gamma'(t) = \mathbf{0}$, then we say that γ is a geodesic. A geodesic γ joining x to y is said to be a minimal geodesic if its length equals $d(p, q)$ and in this case the geodesic γ is called a minimizing geodesic. By the Hopf-Rinow theorem, a

Riemannian manifold \mathcal{M} is complete if and only if any pair of points in \mathcal{M} can be joined by a minimal geodesic. Moreover, if \mathcal{M} is complete, then (\mathcal{M}, d) is a complete metric space and every bounded closed subset is compact.

A Hadamard manifold is a complete, simply connected Riemannian manifold of non-positive sectional curvature. From now on, we always assume that \mathcal{M} is an m -dimensional Hadamard manifold.

The exponential map $\exp_x : \mathcal{T}_x\mathcal{M} \rightarrow \mathcal{M}$ at a point $x \in \mathcal{M}$ is defined by $\exp_x v := \gamma_v(1, x)$ for each $v \in \mathcal{T}_x\mathcal{M}$, where $\gamma(\cdot) := \gamma_v(\cdot, x)$ is the geodesic starting from x with velocity v , i.e., $\gamma(0) = x$ and $\gamma'(0) = v$. It is known that $\exp_x tv = \gamma_v(t, x)$ for any real number t and $\exp_x \mathbf{0} = \gamma_v(0, x) = x$. Note that for $x \in \mathcal{M}$, the exponential map $\exp_x : \mathcal{T}_x\mathcal{M} \rightarrow \mathcal{M}$ is a diffeomorphism. Thus, there exists an inverse exponential map $\exp_x^{-1} : \mathcal{M} \rightarrow \mathcal{T}_x\mathcal{M}$. Moreover, we have $d(x, y) = \|\exp_x^{-1}y\|$ for any $x, y \in \mathcal{M}$.

A geodesic triangle $\Delta(p_1, p_2, p_3)$ is a set consisting of three points p_1, p_2 and p_3 in \mathcal{M} and three minimal geodesics γ_i joining p_i to p_{i+1} , where $i = 1, 2, 3(\text{mod } 3)$.

Proposition 2.1 ([23]). *(Comparison result for triangles). Let $\Delta(p_1 p_2 p_3)$ be a geodesic triangle in \mathcal{M} . For each $i = 1, 2, 3(\text{mod } 3)$, let $\gamma_i : [0, \ell_i] \rightarrow \mathcal{M}$ denote the geodesic joining p_i to p_{i+1} , and $\ell_i = L(\gamma_i)$ and α_i be the angle between tangent vectors $\gamma'_i(0)$ and $p-\gamma'_{i-1}(\ell_{i-1})$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi \tag{2.1}$$

and

$$\ell_i^2 + \ell_{i+1}^2 - 2\ell_i \ell_{i+1} \cos \alpha_{i+1} \leq \ell_{i-1}^2. \tag{2.2}$$

The inequality (2.2) can be rewritten in terms of the distance and the exponential map as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2 \left\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \right\rangle \leq d^2(p_{i+2}, p_i) \tag{2.3}$$

since

$$\left\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \right\rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}.$$

Definition 2.1 ([24]). A subset $\mathcal{K} \subset \mathcal{M}$ is said to be (geodesic) convex if for any two point p and q in \mathcal{K} , the geodesic joining p to q is contained in \mathcal{K} , that is, if $\gamma : [a, b] \rightarrow \mathcal{M}$ is a geodesic such that $\gamma(a) = p$ and $\gamma(b) = q$, then $\gamma(ta + (1 - t)b) \in \mathcal{K}$ for all $t \in [0, 1]$.

Note that for any $x, y \in \mathcal{M}$, there is a unique minimal geodesic γ joining x to y which is defined by $\gamma_{xy}(t) = \exp_x(t \exp_x^{-1} y)$ for all $t \in [0, 1]$ (see, e.g., [22]). Thus, $\mathcal{K} \subset \mathcal{M}$ is a convex set if and only if $\exp_x(t \exp_x^{-1} y) \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and for all $t \in [0, 1]$.

Definition 2.2 ([24]). Let \mathcal{C} be a nonempty convex set of \mathcal{M} . A function $f : \mathcal{C} \rightarrow \mathbb{R}$ is said to be (geodesically) convex if for any geodesic $\gamma : [a, b] \rightarrow \mathcal{C}$ the composition $f \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex.

Equivalently, $f : \mathcal{C} \rightarrow \mathbb{R}$ is convex if and only if

$$f(\exp_x(t \exp_x^{-1} y)) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in \mathcal{C} \text{ and } t \in [0, 1].$$

The distance from a point $x \in \mathcal{M}$ to a subset \mathcal{K} của \mathcal{M} is defined by

$$d(x, \mathcal{K}) := \inf\{d(x, y) : y \in \mathcal{K}\}.$$

The projection of a point $x \in \mathcal{M}$ onto a subset \mathcal{K} of \mathcal{M} is defined by

$$P(x, \mathcal{K}) := \{p \in \mathcal{K} : d(x, p) = d(x, \mathcal{K})\}.$$

Proposition 2.2 ([25]). *Let \mathcal{K} be a closed convex subset of a Hadamard manifold \mathcal{M} . Then, for any $x \in \mathcal{M}$, $P(x, \mathcal{K})$ is a singleton set. Also, for any $p \in \mathcal{M}$, the following assertions are equivalent:*

- (i) $y = P(p, \mathcal{K})$;
- (ii) $\langle \exp_y^{-1} p, \exp_y^{-1} q \rangle \leq 0$ for all $q \in \mathcal{K}$.

For the proof of our main result, we need the following lemma.

Lemma 2.1 ([26]). *Assume that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ then $\lim_{n \rightarrow \infty} a_n$ exists.*

Let \mathcal{X} be a nonempty closed convex subset of \mathcal{M} and $f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in \mathcal{X}$. The *equilibrium problem* on Hadamard manifolds consists of finding an element $x^* \in \mathcal{X}$ such that

$$f(x^*, y) \geq 0 \quad \text{v\o i m\o i } y \in \mathcal{X}. \tag{2.4}$$

In this case, the bifunction f is called an *equilibrium bifunction*. We denote by \mathcal{X}^* the solution set of the equilibrium problem (2.4).

The equilibrium problem on manifolds was first introduced in [8] where the authors pointed out some important problems, which can be formulated by (2.4). In particular, if $f(x, y) = \langle V(x), \exp_x^{-1} y \rangle$ for all $x, y \in \mathcal{X}$, where V is a vector field on \mathcal{M} , then the equilibrium problem (2.4) reduces to the variational inequality problem on Hadamard manifolds introduced by S. Németh ([27]). For results about the existence of solutions of equilibrium problems on manifolds under different assumptions, we refer the reader to [10], [28], [11], [12], [9]. Some methods for solving equilibrium problems on manifolds can be found in [10], [28], [15], [14] and references therein.

Definition 2.3. Let $f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ be a bifunction and \mathcal{X} be a nonempty closed convex subset of \mathcal{M} . The bifunction f is said to be *pseudo-monotone* on \mathcal{X} if, for all $x, y \in \mathcal{X}$,

$$f(x, y) \geq 0 \implies f(y, x) \leq 0.$$

Lemma 2.2. Assume that the solution set \mathcal{X}^* of the equilibrium problem (2.4) is nonempty. Then, \mathcal{X}^* is convex provided that $f(x, \cdot)$ is convex for each $x \in \mathcal{X}$, $f(\cdot, y)$ is upper semicontinuous for each $y \in \mathcal{X}$ and f pseudo-monotone on \mathcal{X} .

Proof. Assume that $x^*, y^* \in \mathcal{X}^*$. We will show that $\gamma_{x^*, y^*}(t) = \exp_{x^*}(t \exp_{x^*}^{-1} y^*) \in \mathcal{X}^*$ for all $t \in (0, 1)$. Let $x \in \mathcal{X}$ and $t \in (0, 1)$ be arbitrary. Since $x^*, y^* \in \mathcal{X}^*$, we have $f(x^*, x) \geq 0$ and $f(y^*, x) \geq 0$. It follows by the pseudo-monotonicity of f that $f(x, x^*) \leq 0$ and $f(x, y^*) \leq 0$. Combining with the convexity of $f(x, \cdot)$ one has

$$f(x, \gamma_{x^*, y^*}(t)) \leq (1 - t)f(x, x^*) + tf(x, y^*) \leq 0.$$

Set $z^* = \gamma_{x^*, y^*}(t)$. We have $f(x, z^*) \leq 0$ for all $x \in \mathcal{X}$. Let y be arbitrary in \mathcal{X} and consider the geodesic $\gamma_{y, z^*}(s) = \exp_y(s \exp_y^{-1} z^*)$ joining y to z^* with $s \in (0, 1)$. By the convexity of $f(\gamma_{y, z^*}(s), \cdot)$, we have

$$\begin{aligned} 0 = f(\gamma_{y, z^*}(s), \gamma_{y, z^*}(s)) &\leq (1 - s)f(\gamma_{y, z^*}(s), y) + sf(\gamma_{y, z^*}(s), z^*) \\ &\leq (1 - s)f(\gamma_{y, z^*}(s), y). \end{aligned}$$

This implies that $f(\exp_y(s \exp_y^{-1} z^*), y) \geq 0$ for all $s \in (0, 1)$. Letting $s \rightarrow 1^-$, one gets $f(z^*, y) \geq 0$. Since $y \in \mathcal{X}$ is arbitrary, $z^* \in \mathcal{X}^*$. That is, $\gamma_{x^*, y^*}(t) \in \mathcal{X}^*$ for all $t \in [0, 1]$. Therefore, the solution set \mathcal{X}^* is convex.

3 Finite convergence of the proximal point method

In this section, we prove the finite convergence of the proximal point method for solving equilibrium problems in Hadamard manifolds under the linear conditioning of the solution set. Our result extends the analogous result in [6] from linear spaces to the manifold setting. In this section, we always assume that the solution set \mathcal{X}^* of the equilibrium problem (2.4) is a nonempty convex subset of \mathcal{M} .

Definition 3.1 ([17], [6]). The solution set \mathcal{X}^* of the equilibrium problem (2.4) is said to be linearly conditioned with modulus $\mu > 0$ if there is a positive constant μ such that

$$-f(x, P_{\mathcal{X}^*}(x)) \geq \mu d(x, \mathcal{X}^*), \quad \text{for all } x \in \mathcal{X}. \tag{3.1}$$

Consider the inexact proximal point method: $x_1 \in \mathcal{X}$ and $\{x_n\}$ satisfies

$$f(x_{n+1}, y) + \eta_n \langle e_{n+1} - \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \quad \text{for all } y \in \mathcal{X}, \tag{3.2}$$

where, $\{e_n\}$ is an error sequence with $e_n \in \mathcal{T}_{x_n} \mathcal{M}$ and $\{\eta_n\} \subset (0, \infty)$ is a sequence of stepsizes.

Note that the method (3.2) is a special case of the inexact proximal point method for solving the problem of finding a singular element of maximal monotone vector fields studied in [29]. Here, we investigate the finite convergence of the method (3.2) when the solution set of considered equilibrium problem satisfies an linearly conditioned assumption and other conditions on $\{e_n\}$ and $\{\eta_n\}$.

Theorem 3.1. *Let $f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ be pseudo-monotone on \mathcal{X} and $\{x_n\}$ be a sequence generated by the inexact proximal point method (3.2). Assume that*

$$\limsup_{n \rightarrow \infty} \eta_n < \infty \quad \text{và} \quad \sum_{n=1}^{\infty} \|e_n\| < \infty.$$

If the solution set \mathcal{X}^ of (2.4) is linearly conditioned with modulus μ , then $x_n \in \mathcal{X}^*$ for all n sufficiently large.*

Proof. Let x^* be a solution of the equilibrium problem (2.4). Then, $f(x^*, x_{n+1}) \geq 0$ for all n . Since f is pseudo-monotone, $f(x_{n+1}, x^*) \leq 0$ for all n . Thus, by (3.2), we have

$$\eta_n \langle e_{n+1} - \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle \geq -f(x_{n+1}, x^*) \geq 0, \text{ for all } n.$$

Since $\eta_n > 0$ với mọi n , $\langle e_{n+1} - \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle \geq 0$. Hence,

$$-\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle + \langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle \geq 0 \text{ for all } n. \quad (3.3)$$

Using (2.3), we have for all n that

$$d^2(x_n, x_{n+1}) + d^2(x_{n+1}, x^*) - 2 \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle \leq d^2(x_n, x^*).$$

It implies that

$$-2 \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle \leq d^2(x_n, x^*) - d^2(x_n, x_{n+1}) - d^2(x_{n+1}, x^*). \quad (3.4)$$

Combining (3.3) and (3.4) we have

$$d^2(x_n, x^*) - d^2(x_{n+1}, x^*) - d^2(x_n, x_{n+1}) + 2 \langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle \geq 0, \quad (3.5)$$

for all n . Thus,

$$d^2(x_{n+1}, x^*) - d^2(x_n, x^*) \leq 2 \langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle \text{ for all } n. \quad (3.6)$$

We next show that

$$d(x_{n+1}, x^*) - d(x_n, x^*) \leq 2 \|e_{n+1}\| \text{ for all } n. \quad (3.7)$$

Indeed, if $\langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle = 0$ for some n , then by (3.6) we have

$$d^2(x_{n+1}, x^*) - d^2(x_n, x^*) \leq 0.$$

Hence,

$$d(x_{n+1}, x^*) - d(x_n, x^*) \leq 0.$$

Assume now that $\langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle \neq 0$. We have $\exp_{x_{n+1}}^{-1} x^* \neq 0$. Thus, $d(x_{n+1}, x^*) \neq 0$. By (3.6), we have

$$[d(x_{n+1}, x^*) - d(x_n, x^*)][d(x_{n+1}, x^*) + d(x_n, x^*)] \leq 2 \|e_{n+1}\| d(x_{n+1}, x^*),$$

which implies that

$$[d(x_{n+1}, x^*) - d(x_n, x^*)][1 + \frac{d(x_n, x^*)}{d(x_{n+1}, x^*)}] \leq 2 \|e_{n+1}\|.$$

This implies that

$$d(x_{n+1}, x^*) - d(x_n, x^*) \leq 2 \|e_{n+1}\|.$$

Thus, (3.7) holds true for all n .

Since $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $d(x_{n+1}, x^*) - d(x_n, x^*) \leq 2 \|e_{n+1}\|$, by Lemma 2.1, $\lim_{n \rightarrow \infty} d(x_n, x^*)$ exists. It means that there exists $\delta \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x^*) = \delta.$$

By (3.5),

$$d^2(x_n, x_{n+1}) \leq d^2(x_n, x^*) - d^2(x_{n+1}, x^*) + 2\langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle \text{ for all } n.$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Since $\limsup_{n \rightarrow \infty} \eta_n < \infty$, $\lim_{n \rightarrow \infty} \|\exp_{x_{n+1}}^{-1} x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|e_n\| = 0$, there exists a natural number n_0 such that

$$\eta_n \|e_{n+1} - \exp_{x_{n+1}}^{-1} x_n\| < \mu, \text{ for all } n \geq n_0. \tag{3.8}$$

For each $n \geq n_0$, by (3.2), we have

$$-f(x_{n+1}, x^*) \leq \eta_n \langle e_{n+1} - \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle, \text{ for all } y \in \mathcal{X}.$$

This implies that

$$-f(x_{n+1}, x^*) \leq \eta_n \|e_{n+1} - \exp_{x_{n+1}}^{-1} x_n\| \cdot \|\exp_{x_{n+1}}^{-1} y\|, \text{ for all } y \in \mathcal{X}.$$

Taking $y := P_{\mathcal{X}^*}(x_{n+1})$, we have

$$\|\exp_{x_{n+1}}^{-1} y\| = \|\exp_{x_{n+1}}^{-1} P_{\mathcal{X}^*}(x_{n+1})\| = d(x_{n+1}, P_{\mathcal{X}^*}(x_{n+1})) = d(x_{n+1}, \mathcal{X}^*).$$

Since \mathcal{X}^* is linearly conditioned with modulus $\mu > 0$,

$$\mu d(x_{n+1}, \mathcal{X}^*) \leq -f(x_{n+1}, P_{\mathcal{X}^*}(x_{n+1})).$$

Thus,

$$\begin{aligned} \mu d(x_{n+1}, \mathcal{X}^*) &\leq \eta_n \|e_{n+1} - \exp_{x_{n+1}}^{-1} x_n\| \cdot \|\exp_{x_{n+1}}^{-1} y\| \\ &= \eta_n \|e_{n+1} - \exp_{x_{n+1}}^{-1} x_n\| d(x_{n+1}, \mathcal{X}^*). \end{aligned} \tag{3.9}$$

If $x_{n+1} \notin \mathcal{X}^*$, then $d(x_{n+1}, \mathcal{X}^*) > 0$. From (3.9) we have

$$\mu \leq \eta_n \|e_{n+1} - \exp_{x_{n+1}}^{-1} x_n\|$$

which contradicts (3.8). Thus, $x_{n+1} \in \mathcal{X}^*$ for all $n \geq n_0$. This ends the proof.

In the above theorem, if we let $e_n = 0$ for all n , then we obtain the finite convergence of sequences generated by the proximal point method.

Theorem 3.2. *Let $\{\eta_n\} \subset (0, a]$ with $a > 0$. Consider the sequence $\{x_n\}$ generated by the following proximal point method: $x_1 \in \mathcal{X}$ and $\{x_n\}$ satisfying*

$$f(x_{n+1}, y) + \eta_n \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \text{ for all } y \in \mathcal{X}. \quad (3.10)$$

If $f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ is pseudo-monotone and \mathcal{X}^ is linearly conditioned with modulus μ , then $\{x_n\}$ converges to a point in \mathcal{X}^* in at most ℓ iterations with*

$$\ell \leq \frac{d^2(x_1, \mathcal{X}^*)a^2}{\mu^2} + 1.$$

Proof. By Theorem 3.1, we have $x_n \in \mathcal{X}^*$ for all n sufficiently large. Let $x^* \in \mathcal{X}^*$. As in the proof of (3.5), one has

$$d^2(x_n, x^*) - d^2(x_{n+1}, x^*) - d^2(x_n, x_{n+1}) + 2\langle e_{n+1}, \exp_{x_{n+1}}^{-1} x^* \rangle \geq 0 \text{ for all } n.$$

Since $e_n = 0$,

$$d^2(x_n, x^*) - d^2(x_{n+1}, x^*) - d^2(x_n, x_{n+1}) \geq 0 \text{ for all } n.$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

For $N \in \mathbb{N}$, we have

$$\begin{aligned} d^2(x_1, x^*) &\geq d^2(x_2, x^*) + d^2(x_1, x_2) \geq \dots \\ &\geq d^2(x_{N+1}, x^*) + \sum_{i=1}^N d^2(x_i, x_{i+1}) \\ &\geq \sum_{i=1}^N d^2(x_i, x_{i+1}). \end{aligned}$$

Thus, for each $N \geq 1$, we have

$$d^2(x_1, \mathcal{X}^*) = \inf_{x^* \in \mathcal{X}^*} d^2(x_1, x^*) \geq \sum_{i=1}^N d^2(x_i, x_{i+1})$$

Let ℓ be the smallest natural number such that $d(x_\ell, x_{\ell+1}) < \mu/a$. We will show that $x_{\ell+1} \in \mathcal{X}^*$. Assume to the contrary that $x_{\ell+1} \notin \mathcal{X}^*$. Set $y_{\ell+1} := P_{\mathcal{X}^*}(x_{\ell+1})$. We have

$$d(x_{\ell+1}, y_{\ell+1}) = d(x_{\ell+1}, \mathcal{X}^*).$$

Since \mathcal{X}^* is linearly conditioned, we have

$$\mu d(x_{\ell+1}, y_{\ell+1}) = \mu d(x_{\ell+1}, \mathcal{X}^*) \leq -f(x_{\ell+1}, P_{\mathcal{X}^*}(x_{\ell+1})) = -f(x_{\ell+1}, y_{\ell+1}). \quad (3.11)$$

By (3.10),

$$\begin{aligned} -f(x_{\ell+1}, y_{\ell+1}) &\leq \eta_\ell \langle \exp_{x_{\ell+1}}^{-1} x_\ell, \exp_{x_{\ell+1}}^{-1} y_{\ell+1} \rangle \\ &\leq \eta_\ell d(x_{\ell+1}, x_\ell) \cdot d(x_{\ell+1}, y_{\ell+1}) \\ &< \eta_\ell \cdot \mu/a \cdot d(x_{\ell+1}, y_{\ell+1}) \leq a \cdot \mu/a \cdot d(x_{\ell+1}, y_{\ell+1}) \\ &= \mu \cdot d(x_{\ell+1}, y_{\ell+1}). \end{aligned} \quad (3.12)$$

Combining (3.11) and (3.12), we get $\mu < \mu$. This is a contradiction. Thus, $x_{\ell+1} \in \mathcal{X}^*$.

Moreover,

$$d^2(x_1, \mathcal{X}^*) = \inf_{x^* \in \mathcal{X}^*} d^2(x_1, x^*) \geq \sum_{i=1}^{\ell-1} d^2(x_i, x_{i+1}) \geq (\ell - 1)\mu^2/a^2.$$

Therefore,

$$\ell \leq \frac{d^2(x_1, \mathcal{X}^*)a^2}{\mu^2} + 1.$$

The proof is complete.

4 Conclusion

In this paper, we have presented some new results for pseudo-monotone equilibrium problems on Hadamard manifolds. We have proved the finite convergence of sequences generated by the proximal point methods when the solution set is linearly conditioned. We have also given an upper bound for the number of iterations by which sequences generated by the proximal point methods converge to a solution of the equilibrium problems. Our results extend analogous results from linear spaces to Hadamard manifolds.

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TÓM TẮT

SỰ HỘI TỤ HỮU HẠN CỦA PHƯƠNG PHÁP ĐIỂM GẦN KÈ GIẢI BÀI TOÁN CÂN BẰNG TRÊN ĐA TẠP HADAMARD

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Trong bài báo này chúng tôi chứng minh sự hội tụ hữu hạn của phương pháp điểm gần kề (xấp xỉ và chính xác) giải bài toán cân bằng giả đơn điệu trên đa tạp Hadamard khi tập nghiệm thỏa mãn trạng thái tuyến tính.

Từ khóa: Bài toán cân bằng; đa tạp Hadamard; song hàm giả đơn điệu; trạng thái tuyến tính; hội tụ hữu hạn; phương pháp điểm gần kề.