A NOTE ON MATCHINGS AND REGULARITY OF GRAPHS

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ABSTRACT

Let G be a simple graph. We introduce the notion of comb-match(G), and prove that $\text{reg}(I(G)) \leq$ comb-match(G)+1. This improves some known bounds.

Keywords: Regularity; edge ideal; matching; Cameron-Walker graph.

1. Introduction

Let $G = (V, E)$ be a finite simple graph with $V = \{1, \ldots, n\}$. The algebra-combinatorics framework used in this paper is described via the edge ideal construction. Let K be a field, and let $R = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring of n variables over K. We associated to G an ideal in R

$$I(G) = \left(x_i x_j \mid \{i, j\} \in E \right) \subset R,$$

which is called the *edge ideal* of G.

The object of our work is the *Castelnuovo-Mumford regularity* (regularity for short) of the edge ideal $I(G)$. Finding bounds for the regularity of $I(G)$ in terms of combinatorial data of G is an active research program in combinatorial commutative algebra in recent years (see e.g. [1], [2], [3], [4], [5], [6], [7], [8]). In this paper we consider some bounds of $\text{reg}(I(G))$ in terms of matchings in G.

A set of edges $M \subseteq E$ of G is called a *matching* if no two edges of M share a common vertex. A vertex is *matched* by M if it is an endpoint of one of the edges in M. A matching M of G is an *induced matching* of G if no two edges of M can connect by an edge of G, i.e. M is of the form $M = \{a_1b_1, \ldots, a_sb_s\}$such that

(1) $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_s\}$ are independent sets of G,

(2) a_ib_j is not an edge of G unless $i = j$

The *matching number* of G, denoted by match(G), is defined by

$$\text{match}(G) = \max\{|M| \mid M \text{ is a matching of } G\},$$

and the *induced matching number* of G, denoted by ind-match(G), is defined by

$$\text{ind-match}(G) = \max\{|M| \mid M \text{ is an induced matching of } G\}.$$

For lower bounds, Katzman [6] showed that

$$\operatorname{reg}(I(G)) \geqslant \text{ind-match}(G) + 1. \tag{1}$$

There are many classes of graphs G for which the equality occurs (see e.g. [9, Theorem 4.12] for the survey).

For upper bounds, Ha and Van Tuyl [4] showed that

$$\operatorname{reg}(I(G)) \leqslant \operatorname{match}(G) + 1. \tag{2}$$

This upper bound is generalized in various ways. We say that a matching $M = \{a_1b_1, \ldots, a_sb_s\}$ is an *ordered matching* if

1. $\{a_1, \ldots, a_s\}$ is an independent set of G,
2. $a_ib_j \in E$ implies $i \leqslant j$.

The *ordered matching number* of G, denoted by $\text{ord-match}(G)$, is defined by

$$\text{ord-match}(G) = \max\{|M| \mid M \text{ is an order matching of } G\}.$$

Obviously, $\text{ord-match}(G) \leqslant \operatorname{match}(G)$. Constantinescu and Varbaro [1] proved that

$$\operatorname{reg}(I(G)) \leqslant \text{ord-match}(G) + 1. \tag{3}$$

A graph G is *chordal* if every its cycle of length at least four has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle, and is *co-chordal* if the complement graph G^c of G is chordal. The *co-chordal cover number*, denoted by $\operatorname{cochord}(G)$, is the minimum number of co-chordal subgraphs required to cover the edges of G. Note that $\operatorname{cochord}(G) \leqslant \operatorname{match}(G)$. Woodroofe [8] proved that

$$\operatorname{reg}(I(G)) \leqslant \operatorname{cochord}(G) + 1. \tag{4}$$

We now define *comb matchings*, which are close to induced matchings.

Definition 1. A matching M of G is called a comb matching if $M = \{a_1b_1, \ldots, a_sb_s\}$ that satisfies:

1. $\{a_1, \ldots, a_s\}$ is an independent set of G,
2. a_ib_j is not an edge of G unless $i = j$.

The *comb matching number* of G is

$$\text{comb-match}(G) = \max\{|M| \mid M \text{ is a comb matching of } G\}.$$

Observe that for a comb matching M of G, the induced subgraph of G on the set of matched vertices by M is a *comb graph* (see Figure 1).

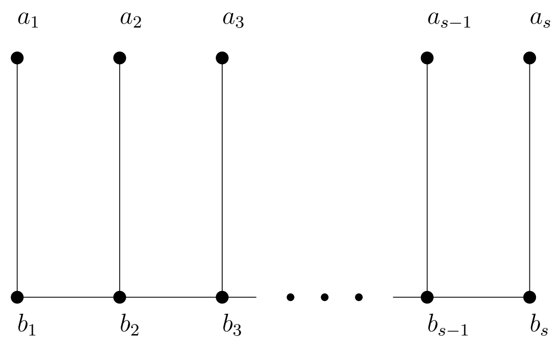

Figure 1: *A comb graph.*

Remark 2. In any graph, we have
Induced matching $\Longrightarrow$ Comb matchings $\Longrightarrow$ Ordered matchings $\Longrightarrow$ Matchings.

The main result of the paper is the following theorem.

Theorem 3. *Let G be a graph. Then,* $\mathrm{reg}(I(G)) \leqslant \text{comb-match}(G) + 1$.

Since $\text{comb-match}(G) \leqslant \text{ord-match}(G)$, the theorem improves the bound (3). Moreover, we also give an example that shows that this bound improves the bound in (4) in some cases.

We explain the structure of this paper. In the next section, we recall some notions and terminology of regularity, simplicial complexes, and graphs. In Section 2, we explicitly compute the comb number matching of paths and cycles. In the last section, we prove the main theorem.

2 Preliminaries

In this section, we first recall the definition of regularity, which can be defined in various ways. For our purpose, we use the definition via the minimal free resolution. Let M be a nonzero finitely generated graded R-module and let

$$0 \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to 0$$

be the minimal free resolution of M. The regularity of M is defined by

$$\mathrm{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

For any nonzero proper homogeneous ideal I of R, by looking at the minimal free resolution, it yields

$$\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1. \tag{5}$$

We next recall some concepts and terminology of simplicial complexes and graphs. A *simplicial complex* Δ over a finite set V is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. Elements of Δ are called *faces*. Maximal faces (with respect to inclusion) are called *facets*. The link of a face σ inside Δ is its subcomplex:

$$\operatorname{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \emptyset\}.$$

Every element in a face of Δ is called a *vertex* of Δ. If there is a vertex, say j, such that $\{j\} \cup \sigma \in \Delta$ for every $\sigma \in \Delta$, then Δ is called a *cone* over j. A complex is called a *simplex* if it contains all subsets of its vertices, and thus a simplex is a cone over every its vertex.

Let Δ be a simplicial complex over the set $V = \{1, \ldots, n\}$. The Stanley-Reisner ideal of Δ is defined to be the squarefree monomial ideal

$$I_\Delta = \left(\prod_{i \in \tau} x_i \mid \tau \subseteq V \text{ and } \tau \notin \Delta\right) \text{ in } R = K[x_1, \ldots, x_n]$$

and the *Stanley-Reisner* ring of Δ to be the quotient ring $k[\Delta] = R/I_\Delta$. This provides a bridge between combinatorics and commutative algebra (see [13], [14]).

Let $G = (V, E)$ be a simple graph. If G has no edges, i.e. $E = \emptyset$, then G is called *trivial*. The *complement* graph G^c of G is the graph whose vertex set is again V and whose edges are the non-edges of G.

When there is no confusion, we simply write uv to also indicate the edge $\{u, v\}$ of G. An edge is incident to a vertex u if $u \in e$. Two vertices u and v are adjacent if $uv \in E$, and in this case, v is a neighbor of u. The set of neighbors of u in G is denoted by $N_G(u)$. For a subset $S \subseteq V$, let

$$N_G[S] = S \cup \{v \in V \mid v \text{ is adjacent to some vertex in } S\}.$$

The degree of a vertex v in G is the number $\deg_G(v) = |N_G(v)|$. The vertex v is called an *isolated* vertex if $\deg_G(v) = 0$, and it is called a *leave* if $\deg_G(v) = 1$.

For a subset S of vertices of G, the *induced subgraph* of G on S, denoted by $G[S]$, is the graph which has vertex set S and edge set consisting of all edges of G with endpoints in S. Denote $G \setminus S$ to be the induced subgraph of G on $V \setminus S$.

A subset S of vertices of G is called an *independent subset* of G if u and v are not adjacent in G for every $u, v \in S$. Let $\Delta(G)$ be the set of independent sets of G. Then it is well-known that $\Delta(G)$ is a simplicial complex, it is the so-called *independence complex* of G. Moreover,

$$I(G) = I_{\Delta(G)}.$$

Note also that for a vertex v of G, we have v is isolated if and only if $\Delta(G)$ is a cone over v; and so $\Delta(G)$ is a simplex if and only if G is trivial. Let $S \in \Delta(G)$, by applying [15, Lemma 2.5] successively, we obtain

$$\operatorname{link}_{\Delta(G)}(S) = \Delta(G \setminus N_G[S]). \tag{6}$$

3 The comb matching number of paths and cycles

In this section we will compute $\mathrm{comb\text{-}match}(G)$ for the case G is ether a path or a cycle. First we define the function $\alpha\colon \mathbb{N} \to \mathbb{N}$ by

$$\alpha(n) = 2k + \left\lfloor \frac{r}{2} \right\rfloor,$$

where $n = 5k + r$ for $k, r \in \mathbb{N}$ with $0 \leqslant r \leqslant 4$. The following properties of this function can be easily verified:

1. $\alpha(n)$ is non-decreasing.
2. $\alpha(n-3) + 1 \leqslant \alpha(n)$ for $n \geqslant 3$.
3. $\alpha(n-5) + 2 = \alpha(n)$ for $n \geqslant 5$.

Next we collect some properties of the comb matching number that we will use in the sequel. If H is a induced subgraph of a graph G, then $\mathrm{comb\text{-}match}(H) \leqslant \mathrm{comb\text{-}match}(G)$; and if G consists of connected components $G_1, \ldots, G_p$, then

$$\mathrm{comb\text{-}match}(G) = \mathrm{comb\text{-}match}(G_1) + \cdots + \mathrm{comb\text{-}match}(G_p).$$

We now compute the comb matching number for paths.

Proposition 4. *Let P_n be the path of n vertices. Then,* $\mathrm{comb\text{-}match}(P_n) = \alpha(n)$.

Proof. Assume that the path P_n is $v_1 v_2 \cdots v_n$. Write $n = 5k + r$ for $k, r \in \mathbb{N}$ where $r \leqslant 4$. For $j = 0, \ldots, k-1$, let Q_j be the path

$$v_{5j+1} v_{5j+2} v_{5j+3} v_{5j+4},$$

and Q is the path $v_{5k+1} \cdots v_{5k+r}$, it has r vertices, $r \leqslant 4$.

Let H be the graph with connected components $Q_0, \ldots, Q_{k-1}, Q$ (see Figure 2). Observe that

$$\mathrm{comb\text{-}match}(Q_i) = 2, \text{ for each } i; \text{ and } \mathrm{comb\text{-}match}(Q) = \left\lfloor \frac{r}{2} \right\rfloor,$$

hence $\mathrm{comb\text{-}match}(H) = \alpha(n)$. Since H is an induced subgraph of G, it follows that

$$\mathrm{comb\text{-}match}(G) \geqslant \mathrm{comb\text{-}match}(H) = \alpha(n).$$

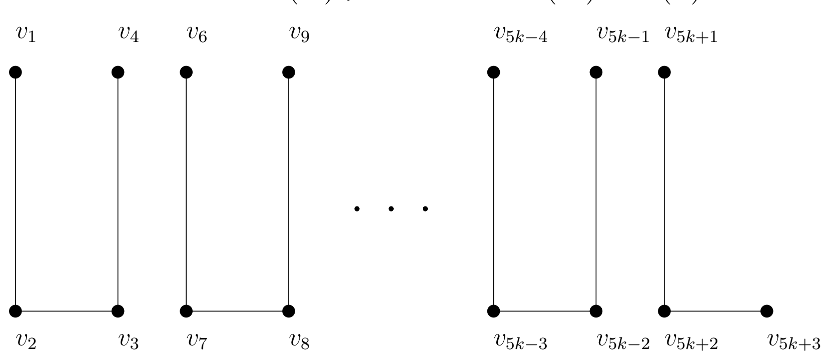

Figure 2: *The graph H*

Therefore, in order to prove the proposition it suffices to show that

$$\text{comb-match}(P_n) \leqslant \alpha(n). \tag{7}$$

We will prove this inequality by induction on n. If $n \leqslant 4$, the inequality is trivial. Assume that $n \geqslant 5$. Let M be a comb matching of P_n such that

$$\text{comb-match}(P_n) = |M|.$$

If v_1 is not matched by M, then M is a comb matching of the path $P = P_n \setminus \{v_1\}$ with $n-1$ vertices, and then $|M| \leqslant \text{comb-match}(P)$. By the induction hypothesis we have

$$\text{comb-match}(P_n) = |M| \leqslant \text{comb-match}(P) \leqslant \alpha(n-1).$$

Note that $\alpha(n-1) \leqslant \alpha(n)$, so that $\text{comb-match}(P_n) \leqslant \alpha(n)$, and the desired inequality holds in this case.

Assume that v_1 is matched by M. Since v_1 is a leave of P_n, it implies that v_1v_2 is an edge in M. We now consider two possible cases:

Case 1: v_3 is matched by M. Since $N_{P_n}(v_3) = \{v_2, v_4\}$ and $v_1v_2 \in M$, $v_3v_4 \in M$. We may assume that $M = \{a_1b_1, \ldots, a_sv_s\}$ satisfies:

1. $\{a_1, \ldots, a_s\}$ is an independent set of G,
2. a_ib_j is not an edge of G unless $i = j$.

Since v_2v_3 is an edge of P_n, one has $v_2, v_3 \in \{b_1, \ldots, b_s\}$, and so $v_4 \in \{a_1, \ldots, a_s\}$. Hence, v_5 is not matched by M as v_4v_5 is an edge of P_n. Let $M' = \{a_3b_3, \ldots, a_sb_s\}$. Then, M' is a comb matching of the path $P' = P_n \setminus \{v_1, \ldots, v_5\}$ with $n-5$ vertices. By the induction hypothesis, $|M'| \leqslant \text{comb-match}(P') \leqslant \alpha(n-5)$. It follows that $\text{comb-match}(P_n) = 2 + |M'| \leqslant 2 + \alpha(n-5) = \alpha(n)$, and the inequality (7) holds.

Case 2: v_3 is not matched by M. In this case, M is a comb matching of the graph $P_n \setminus \{v_3\}$. Since this graph comprises of two connected components that are one edge and a path with $n-3$ vertices, we have

$$|M| \leqslant \text{comb-match}(P_n \setminus \{v_3\}) = 1 + \alpha(n-3) \leqslant \alpha(n).$$

This yields $\text{comb-match}(P_n) \leqslant \alpha(n)$, and the inequality (7) is proved. □

Finally, we compute the comb matching for cycles.

Corollary 5. *Let C_n be the cycle of length n. Then,* $\text{comb-match}(C_n) = \alpha(n-1)$.

Proof. Let $M = \{a_1b_1, \ldots, a_sb_s\}$ be a comb matching of C_n such that

$$\text{comb-match}(C_n) = |M|.$$

By symmetry, we may assume that $a_1 = v_1$ and $a_2 = v_2$. Since $N_G[v_1] = \{v_2, v_n\}$, v_n is not matched by M. Hence, M is a comb matching of the path $P = C_n \setminus \{v_n\}$ with $n-1$ vertices. By Proposition 4 we have

$$\text{comb-match}(C_n) = |M| \leqslant \text{comb-match}(P) = \alpha(n-1).$$

On the other hand, the converse inequality is obvious since P is an induced subgraph of C_n, and the proof is complete. □

4 Proof of the main theorem

This section is devoted to prove Theorem 3. We start by recalling the following result of Hà and Woodroofe.

Lemma 6. *[5, Theorem 5.4] For any simplicial complex* Δ*, we have* $\operatorname{reg}(R/I_\Delta)$ *to be at most the maximum size of a minimal face* σ *with the property that* $\operatorname{link}_\Delta(\sigma)$ *is a simplex.*

We are in position to prove the main result of the paper.

Proof of Theorem 3. Let $G = (V, E)$ be a simple graph. Since $I(G) = I_{\Delta(G)}$, by Lemma 6, there is a face S of $\Delta(G)$ such that

$$\operatorname{reg}(R/I(G)) \leqslant |S|,$$

and S is minimal face with the property that $\operatorname{link}_{\Delta(G)}(S)$ is a simplex.

By Equation (6) we have S is an independent set of G and it is minimal with the property that $G \setminus N_G[S]$ is trivial.

Assume that $S = \{a_1, \dots, a_s\}$. By the minimality of S we have

$$a_i \text{ is not an isolated vertex of } G \setminus N_G[S \setminus \{a_i\}], \text{ for } i = 1, \dots, s.$$

Thus, for each $i = 1, \dots, s$, there is some vertex of the graph $G \setminus N_G[S \setminus \{a_i\}]$, say b_i, such that it is adjacent to a_i. For every $j \neq i$, the set $N_G(a_j)$ of all neighbors of a_j is a subset of $N_G[S \setminus \{a_i\}]$, so that $a_j b_i \notin E$. Hence, the set

$$M = \{a_1 b_1, \dots, a_s b_s\}$$

is a comb matching of G. Together with Equation (5) we obtain

$$\operatorname{reg}(I(G)) = \operatorname{reg}(R/I(G)) + 1 \leqslant s + 1 = |M| + 1 \leqslant \text{comb-match}(G) + 1,$$

and the theorem follows. □

The following example shows that the bound in Theorem 3 is better than the bound in (4) in some cases.

Example 7. Let G be the cycle C_7 of length 7 (see Figure 3). Then, we have

1. $\operatorname{cochord}(G) = \text{ord-match}(G) = \operatorname{match}(G) = 3$.
2. $\text{comb-match}(G) = \text{ind-match}(G) = 2$.

Consequently, $\operatorname{reg}(I(G)) = \text{comb-match}(G) + 1$. Thus, for this graph, the bound in Theorem 3 is better than the bounds (3) and (4).

Proof. Since G has 7 vertices, $\operatorname{match}(G) \leqslant 3$. On the other hand, we have

$$\{\{1, 2\}, \{4, 3\}, \{6, 7\}\}$$

is an ordered matching in G. It implies that $\operatorname{match}(G) = \text{ord-match}(G) = 3$ as $\text{ord-match}(G) \leqslant \operatorname{match}(G)$.

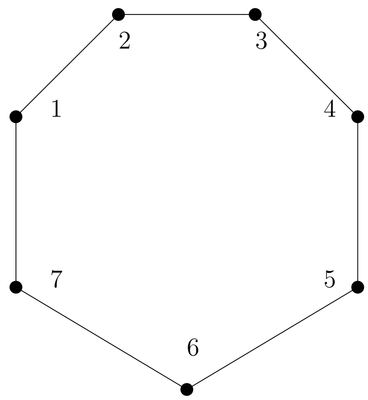

Figure 3: *The cycle C_7*

Note that $\text{comb-match}(G) \leqslant \text{match}(G)$, so $\text{comb-match}(G) \leqslant 3$. Assume that $\text{comb-match}(G) = 3$ and $\{a_1b_1, a_2b_2, a_3b_3\}$ is a comb matching of G. Let v be the remaining vertex of G. Since

$$\deg(a_1) = \deg(a_2) = \deg(a_3) = 2,$$

it follows that all a_i's are adjacent to v. But then v would have degree at least 3, a contradiction. It implies that $\text{comb-match}(G) \leqslant 2$. On the othe hand, since $\{\{1,2\},\{5,6\}\}$ is an induced matching of G, so $\text{comb-match}(G) \geqslant \text{ind-match}(G) \geqslant 2$. Therefore, $\text{comb-match}(G) = \text{ind-match}(G) = 2$.

Finally, we prove that $\text{cochord}(G) = 3$. Since $\text{ind-match}(G) \leqslant \text{cochord}(G) \leqslant \text{match}(G)$, it suffices to show that $\text{cochord}(G) \neq 2$. Now we assume on the contrary that $\text{cochord}(G) = 2$. Then, the edge set E of G could be partitioned into two subsets E_1 and E_2 that corresponding two subgraphs G_1 and G_2 of G, respectively, such that these subgraphs are co-chordal. Since $|E| = 7$, we may assume that $|E_1| \geqslant 4$. Clearly, $G_1 \neq G$ since G is not co-chordal, so G_1 is a forest which consists of paths. Therefore, G_1 is either a path of length at least 4 or containing at least two disjoint paths, hence it has a pairwise disjoint edges which are not connected by some edge of G, and hence it is not co-chordal, a contradiction. It implies that $\text{cochord}(G) = 3$, as required. □

Remark 8. If G is a *Cameron-Walker graph*, i.e. $\text{ind-match}(G) = \text{match}(G)$, (see [10] and [11] for the classification of such a graph), we have

$$\text{ind-match}(G) = \text{comb-match}(G) = \text{cochord}(G) = \text{ord-match}(G) = \text{match}(G),$$

and so $\text{reg}(I(G)) = \text{match}(G) + 1$.

Conversely, if G is a graph with $\text{reg}(I(G)) = \text{match}(G) + 1$, then every its connected component of G is either a Cameron-Walker graph or a pentagon (see [12]).

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TÓM TẮT

MỘT CHÚ Ý VỀ CHỈ SỐ CHÍNH QUY CỦA CÁC ĐỒ THỊ

Đào Thị Thanh Hà

Trường Đại học Sài Gòn, Thành phố Hồ Chí Minh, Việt Nam

Ngày nhận bài 30/12/2024, ngày nhận đăng 12/02/2025

Cho G là một đồ thị đơn, chúng tôi giới thiệu khái niệm comb-match(G), và chứng minh rằng $\operatorname{reg}(I(G)) \leq$ comb-match(G)+1. Kết quả này cải tiến một số kết quả đã biết trước đó.

Từ khóa: Chỉ số chính quy; Iđêan cạnh; cặp ghép; đồ thị Cameron-Walker.