

MEASURES FOR DEVELOPING STUDENTS' REVERSIBLE THINKING COMPETENCE IN TEACHING FUNCTIONS AND GRAPHS IN HIGH SCHOOLS

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Received on 14/7/2022, accepted for publication on 19/9/2022

DOI: <https://doi.org/10.56824/vujs.2022ed10>

Abstract: The reversible thinking is the way of thinking in two opposite directions, which then support each other to help people perceive and solve problems flexibly and effectively. In this article, we mention manifestations of students' reversible thinking competence in teaching functions and graphs and propose measures to foster this ability in order to contribute to improving the quality of teaching mathematics at high schools.

Keywords: Reversible thinking competence; functions and graphs.

1. Introduction

The reversible thinking is a familiar type of thinking related to the observation of things and phenomena in opposite directions but mutually supporting each other and flexibly applied in aspects of life.

Thinking in two opposite directions helps to limit the mistakes of “one-way thinking”. For example, in the field of journalism, if using the mathematical the proof-by-contradictory method to describe information, it can create a high degree of originality and persuasion. Applying the inverse clause structure, the antithesis can also create novel ways of expressing information.

When lecturing the mathematics at high school as well as the functions and graph subjects, the exploitation and application of two-way correlations will help students in dealing with related issues, contribute simultaneously to fostering students' reversible thinking competence.

2. Research content

2.1. *Reversible thinking competence*

In previous mindset research studies, the phrase “reversible thinking” was hardly known in the rest. The reason is that there is no definition that discusses reversible thinking explicitly with its sufficient connotation and denotation. However, the aspects related to reversible thinking have mentioned in research works. J. Piaget had conducted his investigation on the structure of thinking manipulation. The results indicate that the basic nature of the manipulative structures is reversibility, which is the capacity of the human mind to move in both the forward and reverse directions. He argued that the reversibility demonstrated when “manipulations and actions can be deployed in two directions and understanding one of them elicits understanding of the other” (V.V. Davudov, 2000).

In one study, the author V.A. Cruchetsky was interested in the reversibility of the thinking process in mathematical arguments, which can be understood as changing the direction of the thinking process in the sense of moving from forward thinking (from A to B) to inverse thinking (from B to A). Reversible thinking competence was considered as a component of students' mathematical capacities (V.A. Cruchetsky, 1973).

In a book titled "How to solve math problems", G. Polya had referred to the reversible reduction in mathematic problem solving. According to the author, "The conversion from solving the original problem to solving the subproblems is called a reversible or bidirectional reduction, or equivalent if the subproblem and the original problem are approximately the same" (G. Polya, 1997).

With the same interest in the possibility of reversing the thought process, Nguyen Ba Kim, Vu Duong Thuy believes that the formation of opposite thinking in lecturing can be formed simultaneously along with the formation of forward thought, which can be successfully achieved through the parallel practice of forward and reverse transformation capacities (Nguyen Ba Kim, Vu Duong Thuy, 2001). Nguyen Ba Kim took the destination of a fully understood process as the starting point for the new one, while the starting point of the fully understood process was taken the destination of the new one. That was considered by Nguyen Ba Kim as an expression of the versatility of thinking (Nguyen Ba Kim, 2011).

In an author's own publication regarding this issue in 2014, the reversible thinking is the way of thinking in two opposite directions, which then support each other to help people perceive and solve problems flexibly and effectively. Reversible thinking is a specific type of thoughtivity and it always associated with the circumstances which contain matters to be considered in two opposite ways in a certain sense. From that standpoint, reversible thinking competence is a psychological features that reflects different individual finished levels when practicing reversible thinking activities. This capacity is only demonstrated when performing reversible thinking activities and can be cultivated through practice. The development of this capacity depends on the orientation of the lecturers, on the selection of appropriate activities and on the organizing teaching for students to perform those activities.

The above analysis shows that domestic and foreign authors have been interested in studying the reversible thinking matters according to different concepts and objects.

2.2. Manifestations of students' reversible thinking ability in teaching functions and graphs

Regarding the reversible thinking competence and the specification of the contents of function and graphs in high school math programs, when studying functions and graphs in high school, manifestations of reversible thinking of students proposed as follows:

- Identify whether a given curve is a graph of a certain function

A given curve does represent a function if no vertical line can intersect the curve more than once. Otherwise, that curve is not an instance of any graph of the function.

- Be able to determine the properties of functions based on the representation formula, the variation table or the graph of the function; Be able to establish the representation of a function when certain properties of the function are already known

Normally, for a function given by the formula, students will identify the properties of the function, create the variation table and draw the graph of the given function. This can be a manifestation of the forward direction of reversible thinking.

On the contrary, given a table of variation or a graph of a function, students are able to determine the properties of that function. For example, students can find out the evenness of a function by the feature that the graph takes the vertical axis as the axis of symmetry; Students can determine the increasing of the function by the feature that the graph “goes up” from left to right, ... That is, students have the ability to “read” function graphs. From the properties of the function, students can find out the formula for that function. This can be a manifestation of the opposite direction of reversible thinking.

- Be able to build a practical situation that “fits” with the function and vice versa, be able to build compatible functions for practical situations

Each function can have practical situations that the mathematical model of these situations is the function itself. Students are able to determine which practical situations “fit” with the given function. Conversely, mathematical models can also represent practical situation. Students know how to choose a mathematical model, in this case a function, appropriate to each situation. For a better understanding, let us take a look at specific examples: *Given the function $y = 12x$. Indicate practical situations described by the above function.* In this case, let us consider two following cases:

- *Case study 1:* The selling price per kilogram of rice is 12 thousands dong. Thus, the correlation between the returned money y (in thousands dong) and the sold quantity of rice x (in kilograms) can be expressed by the function $y = 12x$.
- *Case study 2:* Taxi rental cost is 12 thousands dong per kilometer. So if the taxi rental distance is x km long, the amount to be paid y (in thousands dong) is calculated according to the formula: $y = 12x$.

- Capable of turning the matter upside down, apply the necessary-and-sufficient problem

Specifically, students have the following symptoms: Set up inversion clauses when learning theorems; Apply necessary conditions, sufficient conditions, necessary and sufficient conditions when learning definitions; Apply necessary conditions, sufficient conditions, necessary and sufficient conditions when learning the topic of functions and graphs, including definitions, properties, and problem-solving methods.

In Mathematics, logical operations, the necessary, sufficient, necessary and sufficient conditions of a clause are frequently used. The understanding of what are necessary and sufficient conditions as well as the identification of problems of necessary and sufficient form, exploiting the mutual correlation between them will help students more convenient in finding ways to solve problems. We go through the following case study, when teaching the theorem: “Assume f reaches its maximum at point x_0 . If f has a derivative at x_0 , then $f'(x_0) = 0$ ” (Doan Quynh, Nguyen Huy Doan, 2010). The reverse may not be true. The derivative f' may be zero at the point x_0 but the function does not have a maximum at x_0 . For example, with the function $f(x) = x^3$. Considering at point $x = 0$, we have $f'(x) = 3x^2$ and $f'(0) = 0$. However, f does not reach its maximum at point $x = 0$, because $f'(x_0) = 0$ for all $x \neq 0$, f is increasing

on R . Thus, the above theorem is only a necessary condition for the function to reach its extreme point. Many students are not aware of this problem, leading to mistakes in solving extreme math problems.

- *Recognizing how to solve a problem by considering the inverse problem or exploiting a two-way correlation between objects*

For example, with the problem of the intersection of two graphs, there are following conclusions: The number of intersections of the graph of function $y = f(x)$ (C_1) and of the function $y = g(x)$ (C_2) is equal to the number of solutions to the equation of the coordinates of their intersection $f(x) = g(x)$ (*). The number of intersections of (C_1) and (C_2) is sometimes determined based on the number of solutions to the equation (*). The number of solutions to the equation (*) (or argument in terms of the number of solutions of (*)) can also be determined, conversely, based on the number of intersections of (C_1) and (C_2). Normally, the equation (*) can be converted to a quadratic equation, while the number of intersections of (C_1) and (C_2) is based on the number of intersections of the line $y = h(m)$ with the same direction as O_x and the curve $y = g(x)$ by isolating parameter. Applying this conclusion means that students already know how to exploit the two-way correlation between to objects, namely the “the number of intersections of the function graph” and “the number of solutions of the coordinate equation”.

- *Evaluating the cognitive and problem solving process*

Re-considering is self-criticism, self-review, re-evaluating the cognitive and problem-solving process which has been self-performed, although there was confidence in the obtained results as well as the methods used previously. This manifested in the ability to question oneself and self-answer the questions such as: Is the result correct or incorrect? Have the cases fully considered? Is there another solution? What would be the result if the problem reversed, enlarged or narrowed?

The review of the cognitive and problem-solving process helps students realize their strong points, shortcomings, fund of knowledge and experiences for timely adjustment and supplementation. Thus, the reviewing process will help students have more solid, deeper and more extensive knowledge, and at the same time, contributes to the development of critical thinking, self-regulation and self-direction for students. In this regard, let us consider the following case: To assert that the function $y = 2x + 1$ is not even, many students prove $f(x) \neq f(-x)$ with $\forall x$ (this is not true when $x = 0$), without knowing that the negation of the statement “true for all values of x ” is the statement “false” with at least one value of x ”. Thus, just pointing out for example $f(1) \neq f(-1)$.

- *Be able to solve mathematical matters in a distinguished, unique solution*

People who have reversible mindset refuses to think in one direction, according to the habit of doing or the opinion of the majority, but also flexibility think in the opposite direction, reconsidering the issue from the opposite angle. This often results in a quick, easy, and unique solution to solve problems. To illustrate this observation, let us consider the following specific case: Find the maximum and minimum value of a function $y = 2x^2 + x - 3$. Typically, the majority of students in grade 10 come up with two solutions:

○ *Solution 1:* Investigate and plot graphs of quadratic functions. The result for the minimum value of the function is $-\frac{25}{8}$ and the function has no maximum value.

○ *Solution 2:* Transform the given function to become: $y = 2x^2 + x - 3 = 2\left(x + \frac{1}{4}\right)^2 - \frac{25}{8} \geq -\frac{25}{8}$ with $\forall x$. From there, determine the minimum value of the function is $-\frac{25}{8}$.

Contrary to the above solutions, by exploiting the meaning of the concept of maximum value and minimum value of a function, students have provided the following solution: Supposing we have to find the extremes of the function $f(x)$ which has a values domain D . Let y be a value of $f(x)$ with $x \in D$ meaning that there is a solution of the equation $f(x) = y$. Finding the conditions for equation $f(x) = y$ to have a solution usually leads to the expression $m \leq y \leq M$. Thence inferred $\min f(x) = m$ with $x \in D$ and $\max f(x) = M$ with $x \in D$.

The detailed solution is as follows: Let y be a value of $f(x)$, we have: $y = 2x^2 + x - 3 \leftrightarrow 2x^2 + x - 3 - y = 0$ (1).

The condition for the quadratic equation (1) to have a solution is: $\Delta \geq 0 \leftrightarrow 8y + 25 \geq 0 \leftrightarrow y \geq -\frac{25}{8}$. So, the minimum value of the function is $-\frac{25}{8}$.

This solution also works for the problem of finding the maximum and minimum value of the function $y = \frac{ax^2+bx+c}{mx+n}$, specifically as follow: The tabulation of variation to find the maximum and minimum value of a function on a given set can only be performed by grade 12 students. However, this matter can be completely solved by students in grade 10 by considering y as a parameter, transforming the given function into a quadratic equation where x is an unknown. By applying the condition of having a solution of the quadratic equation $\Delta \geq 0$, the maximum and minimum values of the function will be determined. Furthermore, by this solution, the existence of the value of x for the function to reach the maximum/minimum value is always asserted.

- Proposing a flexible solution in taking the multiple-choice test

For multiple-choice exercises, students can deduce themselves to choose the most appropriate result. However, with the short duration of the assignment, choosing the right answer for each question is time consuming if only using the essay method, students must be capable in flexible test-taking skills. Let us consider the following multiple-choice question: The function $y = mx^3 - x^2 + (m - 8)x + 1$ is determined to be increased on R if and only if:

- A. $m \in R$ B. $m \leq 0$ C. $m \geq 0$ D. $m \geq \frac{12+7\sqrt{3}}{3}$

In case of following free-response method, starting from the lead sentence, students will come up with steps as follows: Determine the value of m for the function $y = mx^3 - x^2 + (m - 8)x + 1$ to be increase on R . This matter is equivalent to finding m for the derivative of the given function is not negative on R as well as finding m for the following inequality is true: $3mx^2 - 2x + m - 8 \geq 0, \forall x \in R$. After considering the cases $m = 0, m \neq 0$ and find out $m \geq \frac{12+7\sqrt{3}}{3}$, option D will be determined to be correct. The above

steps indicate that the determination of the right choice takes a lot of time, especially in the case of students who have poor skills in transformation and synthesizing solutions.

If the method of exclusion thinking is applied, that is to eliminate the noisy choices among the given alternatives. In the above case, realize that the value $m = 0$ is in all three choices, A, B, and C. Therefore, replacing $m = 0$ into the given function, then there are two possibilities. If the requirements are satisfied, that is, the function f increase, the option D can be eliminated; Otherwise, all three options A, B, and C can be eliminated. Indeed, in this particular case, when $m = 0$, the increment of the given function is not satisfied, so D is the corrected answer. Thus, the direct solution in the forward direction has been simplified by performing the opposite solutions which goes from the answer to evaluate the satisfaction of the matter.

3. Solutions for fostering reversible thinking capacity for students in teaching functions and graphs in high schools

3.1. Practice the skills of “reading, writing and drawing” graphs for students

In teaching functions and graphs at high school, students should be assigned to do the following activities to master the skills of “reading”, “writing”, “drawing” the correct graph of functions, thereby developing their reversible thinking:

- Set up the variation table and draw the graph of the function.
- Set up the formula to represent the function from the graph or from the variation table of the function.
- Determine the function’s properties from the its graph or its variation table.

Take the following case study as an example. Given a function $y = f(x)$, determined on the interval $[-3; 3]$ which is represented as in the following graph.

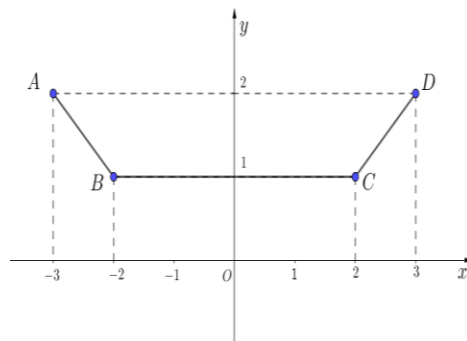


Figure 1: Illustrative examples for solutions to develop reading, writing and drawing skills

Requirement:

- a) Calculate $f(-2)$; $f(3)$; $f(0)$.
- b) Determine the maximum and the minimum values in $[-3; 3]$.
- c) Determine the even and oddness of the function.
- d) Determine the formula for the function.

In this case, the purpose of requirements a, b and c is to practice for students the skills of “reading” from graph of functions. Meanwhile, the requirement d aims to practice the writing skill for students.

3.2. Practice the consideration of the inverse clause and correct application of necessary-and-sufficient matters

- *Practice for students to distinguish between necessary conditions, sufficient conditions, necessary and sufficient conditions*

This solution is illustrated by a specific case study. After studying the following content: “If $f'(x) < 0$ for all x belongs to the interval $(a; b)$, $f(x)$ decrease on the interval $(a; b)$ ” (Tran Van Hao, Vu Tuan, 2008), students are required to find out the condition of parameter a for the function $y = \frac{-1}{3}x^3 + ax^2 + (a - 2)x + 1$ to be decreased on R . This matter actually takes the form of necessary-and-sufficient, meaning that all sufficient conditions (all values of a) for the function to be decreased on R , must be find out. A number of the students gave their answers as follows:

The given function decreases on R , means that $y' < 0, \forall x \in R$

$$\Leftrightarrow -x^2 + 2ax + a - 2 < 0, \forall x \in R$$

$$\Leftrightarrow \Delta' < 0 \Leftrightarrow a^2 + a - 2 < 0$$

$$\Leftrightarrow -2 < a < 1$$

This answer is not correct. Counterexamples can be used to help students discover their own mistakes, thereby self-checking and correcting the solution. Here, the counter-example can be used is the existence of an decrease function when $a = -2$. The following is the correct answer:

The given function decreases on R , means that $y' \leq 0, \forall x \in R$

$$\Leftrightarrow -x^2 + 2ax + a - 2 \leq 0, \forall x \in R$$

$$\Leftrightarrow \Delta' \leq 0$$

$$\Leftrightarrow a^2 + a - 2 \leq 0$$

$$\Leftrightarrow -2 \leq a \leq 1$$

From the above example, to help students in providing the correct solution of these types, teachers need to adjust the knowledge that students receive through the above theorem. This can be through questioning to examine the correctness of the opposite direction of the theorem.

- *Forming a habit of considering the correlation between forward clause, inverse clause, opposite clause and negative clause in studying a theorem to prove it or to discover new knowledge*

To illustrate the solution, let us consider the following case study. After lecturing the definition of a function which a derivative at x_0 , students are assigned to consider the relation between the existence of a derivative and the continuity or the limit of the function at x_0 . Comes from the following forward clause: “If $f(x)$ has a derivative at x_0 then $f(x)$ is continuous at x_0 ”. After suggesting the students to state the inverse clause, opposite clause and the negative clause, the following questions are then posed:

- If $f(x)$ has derivative at x_0 , is it continuous at x_0 ?

- If $f(x)$ is continuous at x_0 , is there a derivative at x_0 ?

- If $f(x)$ has no derivative at x_0 , can it be concluded that it is discontinuous at x_0 ?
- If $f(x)$ is not continuous at x_0 , then how to conclude the derivative at x_0 ?

Students should be encouraged to provide some counterexamples that indicate the wrongfulness of the reverse of the theorem, then answer the above questions. In case of difficulties, counterexamples are given by the teacher and the students are asked to consider the correctness of the statements. The correlation between the continuous, derivative of a function and its graph visualization can also be exploited to help students consider the relationship between the continuity and derivative of a function at x_0 . Then, students can answer the above questions as follows:

- If $f(x)$ has a derivative at x_0 then $f(x)$ is continuous at x_0 . The reverse is incorrect.
- If $f(x)$ is not continuous at x_0 then there is no derivative at x_0 .
- If $f(x)$ has no derivative at x_0 then $f(x)$ can be continuous at x_0 or discontinuous at x_0 .
- The graph of $f(x)$, which has a broken line at x_0 , is discontinuous at x_0 , there is no derivative at x_0 .
- The graph of $f(x)$ is a solid line, so it is continuous at x_0 . Reversely, in case of broken at a point x_0 , $f(x)$ has no derivative at x_0 .

3.3. Practice the skills of using functions to solve practical situations and designing practical situations that fit a given function

After lecturing the quadratic function, students are assigned to solve practical problems, for example: Nam is standing at the footbridge of a three-story flyover in Da Nang City. Knowing that the pylon tower has a parabolic form, the distance of the two pylon towers is about 27 meters which has a height of 20 meters calculated from the point on the ground, which is 2.26 meters away from the foot of the tower (Figure 2). Please help Nam estimate the height of the top of the bridge tower (calculated from the ground) (Ha Huy Khoai, 2021).



Figure 2: Practical situation, solved by the function method

The pylon tower has a parabolic form, so the O_{xy} axis system is chosen, in which one foot of the tower is located at the origin, the other foot is located on O_x . The graph representing the tower, in the form of a parabola, has the following formula: $y = ax^2 +$

bx . To simplify the matter, assuming that each meter length corresponds to a unit in the coordinate system, because the tower is a downward parabola, we choose $a < 0$. From the given data of the issue, the graph of of the parabolic pylon tower as shown in Figure 3.

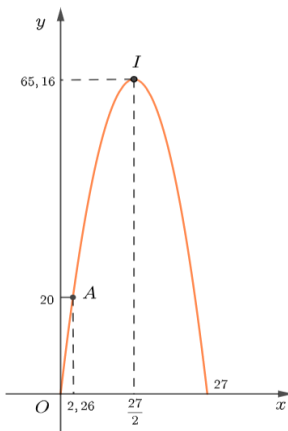


Figure 3: Diagram illustrating the practical problem

Since the distance of the two pylon towers is about 27 meters, there will be two intersection between the graph and the O_x axis, at $x = 0$ and $x = 27$, then we can represent the function as $y = ax(x - 27)$.

According to the issue provided, pylon tower is observed to have a height of 20 meters calculated from the point on the ground, which is 2.26 meters away from the foot of the tower. From there determine the point on the graph, named A, which has a X-coordinate of $x_A = 2,26$. The Y-coordinate of A is $y_A = ax_A(x_A - 27) = 20$, so $20 = a \cdot 2.26 \cdot (2.26 - 27) \leftrightarrow a = -\frac{4000}{11187}$.

Thus, we have the equation representing the pylon tower of the bridge as:

$$y = -\frac{4000}{11187} \cdot x \cdot (x - 27) = -\frac{4000}{11187} \cdot x^2 + \frac{12000}{1243}$$

Notice that the height of the tower is the Y-coordinate of the top of the parabola, which has the coordinate $(\frac{27}{2}; 65.16)$, meaning that the height of the tower is approximately 65 meters.

On the contrary, for a quadric function, students are requested to identify a practical situation which has a mathematical model suitable for that function, for example: Given the function $y = 5 \cdot x^2$ with the the domain $D = [0,5]$. Point out a practical situation in physic that described by the above function.

In order for learners to perform the required activities, the teacher prompts by asking questions about which physical phenomenon is described by the given function? It is necessary to adjust the situation to suit the defined domain of the function. With that pedagogical impact, learners will be associated with the following physical formula $h = \frac{gt^2}{2}$, which describe the distance of a free falling object from the initial position, $g \approx 10 \text{ m/s}^2$. The phenomenon of free falling objects in Physics becomes the focus of attention of learners in the issue of situation building. However, the defined domain of the given function is a matter of concern for the appropriate modification. Obviously, if

the given function describes a free falling object which hits the ground in 5 seconds, meaning that the initial height of the object, compared with the ground, must be 125 meters. Students can make the following statements: *An object is dropped freely from a height of 125 m. The distance y of the object after x seconds from the initial position is $y = 5x^2$; $x \in [0; 5]$.*

The above example is just a specific case of finding a suitable practical situation with a quadratic function. It should be noted that uniformly variable motions in physics, described by formula $S = v_0t + \frac{at^2}{2}$, can be used to design teaching according to the above intention. The above two teaching circumstances have help students approach the concept of quadratic functions from practice, and vice versa, practical application of quadratic functions. And so, it has contributed to fostering reversible thinking ability for students.

4. Conclusion

Competence development for high school students is the goal of the general education program and the Math program in particular, in which the reversible thinking competence plays a key role in forming for students the ability to be flexible and creative in problem detection and problem solving. By describing the manifestations of students' reversible thinking ability in teaching functions and graphs in high schools, we have proposed three pedagogical measures to help teachers meet the requirement of the topic and simultaneously contribute to fostering reversible thinking ability for students.

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TÓM TẮT

MỘT SỐ BIỆN PHÁP PHÁT TRIỂN NĂNG LỰC TƯ DUY THUẬN NGHỊCH CHO HỌC SINH TRONG DẠY HỌC CHỦ ĐỀ HÀM SỐ VÀ ĐỒ THỊ Ở TRƯỜNG TRUNG HỌC PHỔ THÔNG

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Ngày nhận bài 14/7/2022, ngày nhận đăng 19/9/2022

Tư duy thuận nghịch là cách suy nghĩ theo hai chiều hướng trái ngược nhau nhưng tương hỗ lẫn nhau, giúp con người nhận thức và giải quyết vấn đề một cách linh hoạt, hiệu quả. Trong bài báo này, chúng tôi đề cập đến một số biểu hiện của năng lực tư duy thuận nghịch của học sinh trong dạy học chủ đề Hàm số và đồ thị và một số biện pháp bồi dưỡng năng lực này nhằm góp phần nâng cao chất lượng dạy học toán ở trường trung học phổ thông.

Từ khóa: Năng lực tư duy thuận nghịch; chủ đề Hàm số và đồ thị.