UPPER CENTRAL EXPONENT OF LINEAR STOCHASTIC DIFFERENTIAL ALGEBRAIC EQUATIONS OF INDEX 1

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Abstract: In the paper, we introduce the concept of upper central exponent of linear stochastic differential algebraic equation of index 1. We prove that upper central exponent is nonrandom and larger or equal to the top Lyapunov exponent.

Keywords: Lyapunov exponents; central exponents; Stochastic differential equations; Stochastic differential algebraic equations; two-parameter stochastic flow.

1 Introduction

Late in the 19th century, Lyapunov published his profound thesis, which gave birth to the qualitative theory of differential equation. In the thesis, Lyapunov dealt with the long-term behavior of a system of differential equations. He had proposed two methods for studying the systems: Lyapunov exponent method, and the method of Lyapunov functions. Since then, the methods of Lyapunov have been widely used and serve as key tools in investigating the (conditional) stability of the solutions of systems of differential equations. Moreover, developing the method of Lyapunov exponents many researchers introduced various kinds of exponents as further tools for the investigation of qualitative properties of systems of differential equations. The Lyapunov exponents are estimated from above by the central exponents, were introduced in 1957 [1], which are again Lyapunov exponents, only not of the solutions themselves, but of functions obtained from the solutions of the system by applying to these some averaging procedure. A consequence of this procedure is that the central exponents have some properties which the Lyapunov exponents of the solutions themselves do not have. For instance, the negativity of the highest Lyapunov exponent of the linear system $x' = A(t)x$ does not always imply the stability of the trivial solution of the system $x' = A(t)x + g(x,t)$. But the negativity of the upper central exponent of the first system does imply the stability of the trivial solution of the second system.

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The theory of Lyapunov exponents applied to the framework of the ergodic theory leads to a whole new field of research: the theory of random dynamical systems [3]. In [4], author considered the Lyapunov exponents and central exponents of nonautonomous linear stochastic differential equations (SDE) and proved that they are nonrandom. The results that author got based on the tools of the classical theory of Lyapunov developed by Millionshchikov [10], [11] and the theory of two-parameter stochastic flows by Kunita [13]. The central exponents is an upper bound of the Lyapunov exponents.

On the other hand, several technical problems, e.g. in the context of computer-aided design of electronic circuit, the modeling of circuits under the influence of electronic noise leads to the problem of investigating the stochastic differential algebraic equations (SDAE). The class of SDAE of index 1 plays an important role in models with an algebraic component, as e.g. in modified analysis nodal techniques from electrical engineering under the influence of inner electrical noise (see [7]). This comes from the fact that the technique of modified analysis nodal without noise yields a DAE of index 1 if and only if the network contains neither CV-loops nor LI-cutsets (see [12]) and under some additional conditions, the noise sources do not appear in the constraints (see [9], [14], [15]). For a SDAE of index 1 under certain conditions, we are able to transform it into a system consisting of a SDE and an algebraic equations (see [5]). So that we may use methods and results of the theory of SDE to them. In [5] and [7], the authors introduced the concept of Lyapunov exponents, Lyapunov spectrum and Lyapunov regularity of a SDAE of index 1.

The paper is organized as follows: in the next section we recall some basic notions about two-parameter stochastic flows, the Lyapunov exponents and central exponents of linear SDE; the induced two-parameter stochastic flows, Lyapunov exponents and the Lyapunov spectrum of SDAE of index 1. In Section 3 we introduce the concept of central exponent of linear SDAE of index 1 and prove that it is nonrandom and compare with top Lyapunov exponent.

The following notations are used throughout the paper: $\mathcal{G}_k$ is the Grassmannian manifold of all k-dimensional subspaces of $\mathbb{R}^n$; U_* is the subset of all nonvanishing vectors of a linear subspace $U \subset \mathbb{R}^n$; $\|.\|$ is the standard both Euclidean norm on $\mathbb{R}^n$ and operator norm, $\mathbb{T}|_U$ denotes the restriction of the operator $\mathbb{T}$ on U with operator norm $\|\mathbb{T}|_U\| = \sup_{x\in U_*} \frac{\|\mathbb{T}(x)\|}{\|x\|}$.

2 Preliminary

2.1 Stochastic differential equation and two-parameter stochastic flow

Consider the nonautonomous linear SDE

$$dx(t) = F_0(t)x(t)dt + \sum_{j=1}^{d} F_j(t)x(t)dW_t^j, \quad t \in \mathbb{R}^+, \tag{2.1}$$

where the matrix-valued functions $F_j : \mathbb{R}^+ \to \mathbb{R}^{n\times n}, j = 0, \dots, d$, are continuous and (W_t) denotes an d-dimensional Wiener process given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see [2]).

Kunita [13] has introduced a concept of two-parameter stochastic flows which is a dynamic approach to the theory of SDEs since under mild conditions an SDE generates a two-parameter stochastic flow $\Phi_{s,t}(\omega)$ of linear operators of $\mathbb{R}^n$. Moreover, the solution of (2.1), satisfying initial value condition $x(0) = x_0$, is a stochastic process given by the formula $x(t) = \Phi_{s,t}(\omega)x_0$. Conversely, under a reasonable condition a two-parameter flow of diffeomorphisms of R^n generates an SDE. Therefore, there is a one-to-one correspondence between smooth SDEs and smooth two-parameter stochastic flows. Now, we recall the concept of two-parameter stochastic flow on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathbb{R}^n$. A two-parameter stochastic flow of diffeomorphisms of $\mathbb{R}^n$ is a family of continuous maps $\{\Phi_{s,t}(\omega) : \omega \in \Omega, s, t \in \mathbb{R}^+\}$ which satisfies the following properties for almost surely

(i) $\Phi_{s,t}(\omega) = \Phi_{u,t} \circ \Phi_{s,u}(\omega)$ holds for all $s, t, u \in \mathbb{R}^+$, where $\circ$ denotes the composition of maps;

(ii) $\Phi_{s,s}(\omega)$ is the identity map for all $s \in \mathbb{R}^+$;

(iii) the map $\Phi_{s,t}(\omega) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is an onto homeomorphism for all $s, t \in \mathbb{R}^+$;

(iv) $\Phi_{s,t}(\omega)x$ is differentiable with respect to $x \in \mathbb{R}^n$ for all $s, t \in \mathbb{R}^+$ and the derivative is continuous in s, t, x.

We shall assume that the properties $(i) - (iv)$ above are satisfied for all $\omega \in \Omega$. Millionshchikov [10], [11] had discovered that there are several equivalent definitions for the Lyapunov spectrum. Nguyen Dinh Cong used following definition [4]: for a given two-parameter stochastic flow $\Phi_{s,t}(\omega)$ of linear operator of $\mathbb{R}^n$, the extended-real numbers

$$\lambda_k(\omega) := \inf_{V\in\mathcal{G}_{n-k+1}} \sup_{x\in V} \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_{0,t}(\omega)x\|, \quad k = 1, ..., n,$$

are called Lyapunov exponents of the flow $\Phi_{s,t}(\omega)$. The set consisting of the Lyapunov exponents $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_n(\omega)$ is called Lyapunov spectrum of the flow $\Phi_{s,t}(\omega)$. The extended-real numbers

$$\Omega_k(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \inf_{T \in \mathbb{R}^+} \limsup_{m \to \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)V}\|, k = 1, 2, \ldots, n$$

are called the central exponents of the two-parameter flow $\Phi_{s,t}(\omega)$. The Lyapunov exponents and Lyapunov spectrum, central exponents of (2.1) are, by definition, those of two-parameter stochastic flow $\Phi_{s,t}(\omega)$ generated by (2.1). It was proved that Lyapunov exponents, central exponents of (2.1) are nonrandom (see [4], [8]).

2.2 Lyapunov exponents of stochastic differential algebraic equation

Let us consider the linear SDAE of index 1

$$A(t)dx(t) + B(t)x(t)dt + \sum_{j=1}^{d} B_j(t)x(t)dW_t^j = 0, \; t \in \mathbb{R}^+, \tag{2.2}$$

where, matrix-valued functions $A, B, B_j : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ are continuous and (W_t) denotes an d-dimensional Wiener process.

We assume that $A(t)$ is singular with $\operatorname{rank} A(t) = r < n$ and smooth kernel $\ker A(t)$, i.e. there exists a smooth projector $Q \in C^1(\mathbb{R}^+, \mathbb{R}^{n \times n})$ onto $\ker A(t)$. Recall that the SDAE (2.2) is said to be of *index 1* [5], if

(i) the deterministic part of (2.2) is a DAE of index 1, i.e. $A_1(t) := A(t) + B_0(t)Q(t)$ is nonsingular on $\mathbb{R}^+$, where $B_0(t) := B(t) - A(t)P'(t), P(t) := I - Q(t)$;

(ii) $B_j(t)x \subset A(t)$ for all $(t, x) \in \mathbb{R}^+ \times$ and $j = 1, \ldots, d$.

If the SDAE (2.2) is of index 1, then the *inherent SDE* of (2.2) (under P) is (see [5], [6])

$$du(t) = (P' - PA_1^{-1}B_0)u(t)dt + \sum_{j=1}^{d} F_j(t)u(t)dW_t^j, \; t \in \mathbb{R}^+, \tag{2.3}$$

where $F_j(t) := -A_1^{-1}(t)B_j(t)P_{can}(t), \; j = 1, \ldots, d$. Denote $Q_{can} := QA_1^{-1}B_0$ and $P_{can} := I - Q_{can}$. Note that Q_{can} is a projector onto $\ker A$ which is independent of the choice the projector Q.

Suppose that $\Phi_{s,t}(\omega)$ is a two-parameter stochastic flow generated by (2.3). Then, $x(t) = P_{can}(t)\Phi_{s,t}(\omega)P(s)x_0, \; x_0 \in \mathbb{R}^n$, is the solution of (2.2) satisfying the initial condition $x(s) - x_0 \in \ker A(s), \; t \geq s \geq 0$. As in [6] we call $\Psi_{s,t}(\omega) := P_{can}(t)\Phi_{s,t}(\omega)P(s)$

the induced two-parameter flow of (2.2). For any $x \in \mathbb{R}^n$, $s, t \in \mathbb{R}^+$, we have

$$\begin{aligned} \Psi_{0,t}(\omega)x &= P_{can}(t)\Phi_{0,t}(\omega)P(0)x \\ &= P_{can}(t)\Phi_{s,t}(\omega)\Phi_{0,s}(\omega)P(0)x \\ &= \Psi_{s,t}(\omega)\Psi_{0,s}(\omega)x. \end{aligned} \tag{2.4}$$

Following the ideas in references [4], [11], the Lyapunov exponents of an SDAE (2.2) of index 1 are defined by the formula (see [6])

$$\lambda_k(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \sup_{x \in V} \limsup_{t \to \infty} \frac{1}{t} \log \|\Psi_{0,t}(\omega)x\|, \; k = 1, \dots, r. \tag{2.5}$$

The set consisting of the Lyapunov exponents $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_r(\omega)$ is called *Lyapunov spectrum* of (2.2). Lyapunov exponents are measurable and are non-random.

3 Upper central exponent of linear stochastic differential algebraic equation of index 1

In this section we consider nonautonomous linear SDAE (2.2) of index 1. Let $\Psi_{s,t}(\omega) = P_{can}(t)\Phi_{s,t}(\omega)P(s)$ be the induced two-parameter flow of (2.2), where, $\Phi_{s,t}(\omega)$ is a two-parameter stochastic flow generated by inherent SDE (2.3).

Using the approach of references [4], [6], we introduce the notion of upper central exponent for (2.2).

Definition 3.1. *Let $\Psi_{s,t}(\omega)$ be the induced two-parameter flow of (2.2). Then*

$$\Lambda(\omega) := \inf_{T \in \mathbb{R}^+} \limsup_{\substack{m \in \mathbb{N} \\ m \to \infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\| \tag{3.1}$$

is called the upper central exponent of the two-parameter flow $\Psi_{s,t}(\omega)$. The upper central exponent of SDAE (2.2) is, by definition, the upper central exponent of the induced two-parameter stochastic flow generated by the SDAE.

Theorem 3.2. *The upper central exponent $\Lambda(\omega)$ of SDAE (2.2) is always greater than or equal to the top Lyapunov exponent $\lambda_1(\omega)$.*

Note that already in the deterministic case the above relation can be strict for an example of a two-dimensional deterministic system (see [1]).

Proof. Acording to (2.5), we have the top Lyapunov exponent of SDAE (2.2)

$$\lambda_1(\omega) = \sup_{x\in\mathbb{R}^n} \limsup_{t\to\infty} \frac{1}{t} \log \|\Psi_{0,t}(\omega)x\|.$$

It can be computed via a discrete-time interpolation of the flow: for any $T > 0$,

$$\lambda_1(\omega) = \sup_{x\in\mathbb{R}^n} \limsup_{\substack{m\in\mathbb{N}_*\\ m\to\infty}} \frac{1}{mT} \log \|\Psi_{0,mT}(\omega)x\|. \tag{3.2}$$

For any $T > 0$, $m \in \mathbb{N}_*$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\Psi_{0,mT}(\omega)x\| &\overset{(2.4)}{=} \|\Psi_{(m-1)T,mT}(\omega)\Psi_{0,(m-1)T}(\omega)x\| \\ &\leq \|\Psi_{(m-1)T,mT}(\omega)|_{\Psi_{0,(m-1)T}(\omega)\mathbb{R}^n}\|\|\Psi_{0,(m-1)T}(\omega)x\| \\ &\leq \|\Psi_{(m-1)T,mT}(\omega)|_{\Psi_{0,(m-1)T}(\omega)\mathbb{R}^n}\| \dots \|\Psi_{0,T}(\omega)|_{\mathbb{R}^n}\|\|x\|. \end{aligned}$$

Therefore, for any $T > 0$ and $m \in \mathbb{N}_*$

$$\frac{1}{mT} \log \|\Psi_{0,mT}(\omega)x\| \leq \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\| + \log \|x\|.$$

From this inequality and (3.2), we get for any fixed $T > 0$,

$$\lambda_1(\omega) = \sup_{x\in\mathbb{R}^n} \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT} \log \|\Psi_{0,mT}(\omega)x\| \leq \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\|.$$

Consequently,

$$\lambda_1(\omega) \leq \inf_{T>0} \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\| \overset{(3.1)}{=} \Lambda(\omega).$$

The theorem is proved. □

Theorem 3.3. *The upper central exponent $\Lambda(\omega)$ of SDAE (2.2) is not random.*

Proof. For $N \in \mathbb{N}_*$, and $T > 0$, denote $\xi_N := \sum_{i=0}^{N} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\|$. Then the random variable ξ_N has finite second moment. This implies

$$\limsup_{m\to\infty} \frac{1}{mT}\xi_N = 0, \text{ almots surely.}$$

Therefore, for any $N \in \mathbb{N}_*$

$$\begin{aligned}\Lambda(\omega) &= \inf_{T>0} \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT}\big(\xi_N + \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\|\big)\\ &= \inf_{T>0} \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT}\sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\|\\ &= \inf_{T>0} \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT}\sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{NT,iT}(\Phi_{0,NT}(\omega)P(0)\mathbb{R}^n)}\|\\ &= \inf_{T>0} \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT}\sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{NT,iT}(\omega)\mathbb{R}^n}\|.\end{aligned}$$

Now, for $N \in \mathbb{N}_*, T > 0$, put

$$f(T,N,\omega) := \limsup_{\substack{m\in\mathbb{N}\\ m\to\infty}} \frac{1}{mT}\sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{NT,iT}(\omega)\mathbb{R}^n}\|.$$

Similar to [8], we have

$$\inf_{T>0} f(T,N,\omega) = \inf_{T\in\mathbb{N}_*} f(T,N,\omega).$$

Denote $\mathcal{F}_s^t := \sigma(W_u - W_s, 0 \le s \le u \le t)$. Note that $\mathcal{F}_t = \mathcal{F}_0^t$, where $\mathcal{F}_t := \sigma(W_s, 0 \le s \le t), t \in \mathbb{R}^+$, is the natural filtration of Brownian motion $(W_t)_{t\ge 0}$. Then, $f(T,N,\omega)$ is $\mathcal{F}_{NT}^\infty$-measurable, since $\Psi_{s,t}(\omega) = P_{can}(t)\Phi_{s,t}(\omega)P(s)$, $t > s \ge 0$, is $\mathcal{F}_s^t$-measurable. Therefore, for any $N \in \mathbb{N}_*$ the random variable $\Lambda(\omega)$ is measurable with respect to the σ-algebra $\bigcap_{T\in\mathbb{N}_*} \mathcal{F}_{NT}^\infty$. Hence $\Lambda(\omega)$ is measurable with respect to the tail σ-algebra $\bigcap_{N\in\mathbb{N}_*}\bigcap_{T\in\mathbb{N}_*} \mathcal{F}_{NT}^\infty$. Since Brownian motion has independent increments, the tail σ-algebra $\bigcap_{N\in\mathbb{N}_*}\bigcap_{T\in\mathbb{N}_*} \mathcal{F}_{NT}^\infty$ is trivial by Kolmogorov's zero-one law. Consequently, the random variable $\Lambda(\omega)$ is degenerate, i.e., nonrandom. The theorem is proved. □

Theorem 3.4. *Let (2.2) be an SDAE of index 1 with coefficients $A(t)$, $B(t)$, A_1^{-1}, $P'(t)$, $B_i(t), i = 1, ..., d$ are bounded. Then the central exponent $\Lambda(\omega)$ of SDAE (2.2) is smaller than or equal to the central exponent of the corresponding inherent SDE (2.3).*

Proof. Let $\Phi_{s,t}(\omega)$ be the two-parameter stochastic flow of the inherent linear SDE (2.3) and $\Psi_{s,t}(\omega) = P_{can}(t)\Phi_{s,t}(\omega)P(s)$ is the induced two-parameter flow of (2.2). For any $t \ge s \ge 0$, we have

$$\Psi_{s,t}(\omega) = P_{can}(t)\Phi_{s,t}(\omega)P(s).$$

Hence,

$$\|\Psi_{s,t}(\omega)|_{\Psi_{0,s}(\omega)\mathbb{R}^n}\| \leq \|\Psi_{s,t}(\omega)\| \leq \|P_{can}(t)\|\|\Phi_{s,t}(\omega)\|\|P(s)\|.$$

Consequently,

$$\log\|\Psi_{s,t}(\omega)|_{\Psi_{0,s}(\omega)\mathbb{R}^n}\| \leq \log\|P_{can}(t)\| + \log\|\Phi_{s,t}(\omega)\| + \log\|P(s)\|.$$

This implies that for any $T > 0, m \in \mathbb{N}_*$

$$\sum_{i=0}^{m-1} \log\|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\| \leq m(\log\|P_{can}(t)\| + \log\|P(s)\|) + \sum_{i=0}^{m-1} \log\|\Phi_{iT,(i+1)T}(\omega)\|.$$

From the condition on bounded of P_{can}, P and the relations $P = A_1^{-1}A$, $Q = I - P$, $P_{can} = I - QA_1^{-1}(B - AP')$ it follows that

$$\inf_{T\in\mathbb{R}^+} \limsup_{\substack{m\in\mathbb{N}_* \\ m\to\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \log\|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)\mathbb{R}^n}\| \leq \inf_{T\in\mathbb{R}^+} \limsup_{\substack{m\in\mathbb{N}_* \\ m\to\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \log\|\Phi_{iT,(i+1)T}(\omega)\|.$$

The limit in the right hand side of this inequality is the upper central exponent of SDE (2.3) and the limit in the left hand side is the upper central exponent of SDAE (2.2). The theorem is proved. □

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TÓM TẮT

SỐ MŨ TRUNG TÂM TRÊN CỦA PHƯƠNG TRÌNH VI PHÂN ĐẠI SỐ NGẪU NHIÊN TUYẾN TÍNH CHỈ SỐ 1

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Ngày nhận bài 12/10/2022, ngày nhận đăng 21/11/2022

Trong bài báo này, chúng tôi giới thiệu khái niệm số mũ trung tâm trên cho phương trình vi phân đại số tuyến tính chỉ số 1. Chúng tôi chứng minh rằng đối với phương trình vi phân đại số ngẫu nhiên tuyến tính chỉ số 1 thì số mũ trung tâm trên là không ngẫu nhiên và lớn hơn hoặc bằng số mũ Lyapunov lớn nhất.

Từ khóa: Số mũ Lyapunov; số mũ trung tâm; phương trình vi phân ngẫu nhiên; phương trình vi phân đại số ngẫu nhiên; dòng ngẫu nhiên hai tham số.