ON CONSTANT RANK-TYPE CONSTRAINT QUALIFICATIONS FOR SECOND ORDER CONE PROGRAMS IN TWO-DIMENSIONAL SPACES

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ABSTRACT
In this paper, we study the relationship between the constant rank-type constraint qualifications for nonlinear second-order cone programs, which was recently developed by R. Andreani et al. ([2], [3]) in two-dimensional space.

Keywords: Constant rank-type constraint qualifications; CRCQ; weak-CRCQ; seq-CRCQ; second-order cone programming.

1. Introduction

In the classical nonlinear programming (NLP) context, the constant rank constraint qualification (CRCQ) was introduced by Janin [6], with the purpose of obtaining a formula for the Hadamard directional derivative of the value function. Moreover, there were expansions and development of its applications.

Although constraint qualifications with applications toward convergence of algorithms are largely studied in NLP, the situation is quite different in nonlinear second-order cone programming (NSOCP), despite its many relevant applications-for example, in structural optimization and machine learning, hydroacoustic classification of fishes, and others. In NSOCP, this role is almost always covered by the so-called nondegeneracy condition and Robinson's constraint qualification, which can be seen as natural generalizations of LICQ and MFCQ, respectively.

In 2019, Zhang and Zhang ([7]) first proposed extending the CRCQ to the NSOCP, but turns out that their results were incorrect ([1]). Later, R. Andreani et al. developed some constant rank-type constraint qualifications for the NSOCP ([2], [3]). In [3], the authors introduced the weak constant rank constraint qualification (weak-CRCQ), the sequential constant rank constraint qualification (seq-CRCQ) and its applications. The constant rank constraint qualification (CRCQ) was introduced in [2]. Furthermore, they est ablished the following relationship between these constraint qualifications: seq-CRCQ implies weak-CRCQ; and in the general case, neither CRCQ nor weak-CRCQ implies seq-CRCQ. In this paper, we study the relationship between the above constraint qualifications for the NSOCP in the case of single constraint in two-dimensional space and compare with CRCQ for the NLP in this case.

2 Preliminaries

In this paper, all spaces are assumed to be Euclidean spaces with scalar product $\langle ., . \rangle$ Euclidean norm $\|.\|$. For a given set $S \subseteq \mathbb{R}^n$ we will denote the polar of S by

$$S^o = \{u \in \mathbb{R}^n | \langle u, x\rangle \leq 0, \text{ for all } x \in S$$

and the orthogonal complement of S will be denoted by $S^{\perp}$. The notations $\text{int}(S), \text{bd}(S)$, and $\text{bd}^+(S)$ stand for the topological interior, boundary, and boundary excluding the origin of S in $\mathbb{R}^n$, respectively. For a given set $S \subseteq \mathbb{N}$ we denote the number of elements in the set s by $|S|$.

Let Ω be a nonempty subset of the Euclidean space $\mathbb{R}^n$ and $\bar{x}$ be a point in Ω. The (Bouligand-Severi) *tangent/contingent cone* to the set Ω at $\bar{x} \in \Omega$ is known as

$$T_\Omega(\bar{x}) := \big\{v \in \mathbb{R}^n | \text{ there exist } t_k \downarrow 0, \ v_k \to v \ \text{ with } \ \bar{x} + t_k v_k \in \Omega \text{ for all } k \in \mathbb{N}\big\}.$$

The polar cone of the tangent cone is the (Fréchet) *regular normal cone* to Ω at $\bar{x}$ defined by

$$\widehat{N}_\Omega(\bar{x}) := T_\Omega(\bar{x})^\circ. \tag{1}$$

It is well-known that the regular normal cone could be presented by the following construction

$$\widehat{N}_\Omega(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. Another normal cone construction used in our work is the (Mordukhovich) *limiting/basic normal cone* to Ω at $\bar{x} \in \Omega$ defined by

$$N_\Omega(\bar{x}) = \big\{v \in \mathbb{R}^n \mid \text{there exist } x_k \xrightarrow{\Omega} \bar{x}, \ v_k \in \widehat{N}_\Omega(x_k) \text{ with } v_k \to v\big\}.$$

If $\bar{x} \notin \Omega$, one puts $T_\Omega(\bar{x}) = \emptyset$ and $N_\Omega(\bar{x}) = \widehat{N}_\Omega(\bar{x}) = \emptyset$ by convention. When the set Ω is convex, the above tangent cone and normal cones reduce to the tangent cone and normal cone in the sense of convex analysis.

Consider the following nonlinear second-order cone programming (NSOCP) problem:

$$\begin{array}{lll} \text{Minimize} & f(x) & \\ \text{s.t.} & g_j(x) \in K_{m_j}, \ \ j = 1, 2, ..., q, & \end{array} \qquad \text{(NSOCP)} \tag{2}$$

where $K_{m_j} := \{(z_0, \widehat{z}) \in \mathbb{R} \times \mathbb{R}^{m_j - 1} | z_0 \geq \|\widehat{z}\|\}$ when $m_j > 1$ and $K_1 = \mathbb{R}_+$; $f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$, $\ j = 1, 2, ..., q$ are twice continuously differentiable.
Let $\mathcal{F} := \{x \in \mathbb{R}^n | g_j(x) \in K_{m_j}, \ \ j = 1, 2, ..., q.\}$ be the feasible set of (NSOCP). Given a feasible point $\bar{x} \in \mathcal{F}$, let us define the following index sets:

$$I_{\text{int}}(\bar{x}) := \{j \in \{1, 2, ..., q\} | g_j(\bar{x}) \in \text{int}(K_{m_j})\},$$

$$I_B(\bar{x}) := \{j \in \{1, 2, ..., q\} | g_j(\bar{x}) \in \mathrm{bd}^+(K_{m_j})\},$$
$$I_0(\bar{x}) := \{j \in \{1, 2, ..., q\} | g_j(\bar{x}) = 0\}.$$

Denote

$$\mathcal{C} := \prod_{j \in I_0(\bar{x})} K_{m_j} \times \mathbb{R}_+^{|I_B(\bar{x})|},$$
$$\mathcal{G}(x) := (\mathcal{G}_j(x))_{j \in I_0(\bar{x}) \cup I_B(\bar{x})},$$
$$\mathcal{G}_j(x) := \begin{cases} g_j(x) & \text{if } j \in I_0(\bar{x}), \\ \phi(x) & \text{if } j \in I_B(\bar{x}), \end{cases}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^{|I_B(\bar{x})|}$ has its j-th component given by

$$\phi_j(x) := [g_j(x)]_0 - \|\widehat{g_j(x)}\|.$$

We recall that F is a *face* of $\mathcal{C}$ if every open line segment that contains a point of F also has its extrema in F; that is, if for every $y \in F$ and every $z, w \in \mathcal{C}$ such that $y = az + (1-a)w$ for some $a \in (0, 1)$, we have that $z, w \in F$. Further, when there exists some $\eta \in \mathcal{C}^o$ such that

$$F = \mathcal{C} \cap \{\eta\}^\perp,$$

that is, when F is the intersection between $\mathcal{C}$ and one of its supporting hyperplanes, we say that F is an *exposed face* of $\mathcal{C}$. We use the notation $F \trianglerighteq \mathcal{C}$ to say that F is a face of $\mathcal{C}$.

For each $j = 1, 2, ..., q$ the cone K_{m_j} is facially exposed, meaning all of their faces are exposed. Hence, $F_j \trianglerighteq K_{m_j}$ are limited to only three types:

♦ The vertex, $\{0\}$, which can be characterized by any $\eta \in \mathrm{int} K_{m_j}$;

♦ The cone K_{m_j} itself, which is characterized by $\eta = 0$;

♦ A ray at the boundary of K_{m_j} , starting at the vertex and passing through a point $z = (z_0, \widehat{z}) \in \mathrm{bd}^+(K_{m_j})$, which can be characterized by any $\eta \in \mathrm{cone}(z_0, -\widehat{z}) \backslash \{0\}$.

Definition 2.1 ([2, Definition 4.1]) Let $\bar{x}$ be a feasible point of (NSOCP). We say that the *facial constant rank property* (FCRP) holds at $\bar{x}$ if there exists a neighborhood $\mathcal{V}$ of $\bar{x}$ such that for each $F \trianglerighteq \mathcal{C}$, the dimension of $D\mathcal{G}(x)^T[F^\perp]$ is constant for $x \in \mathcal{V}$.

For $\bar{x} \in \mathcal{F}$, set $H(\bar{x})$ defined by

$$H(\bar{x}) := Dg(\bar{x})^T N_{\mathcal{K}}(g(\bar{x})) = \Big\{Dg(\bar{x})^T z | z \in N_{\mathcal{K}}(g(\bar{x}))\Big\}.$$

Definition 2.2 ([2, Definition 4.2]) Let $\bar{x}$ be a feasible point of (NSOCP). We say that the *constant rank constraint qualification* (CRCQ) for NSOCP holds at $\bar{x}$, if it satisfies the facial constant rank property and, in addition, the set $H(\bar{x})$ is closed.

By [2, Theorem 4.2], we see that CRCQ implies Abadie's CQ. Moreover, it is a constraint qualification strictly weaker than nondegeneracy and independent of Robinson's CQ.

Remark 2.3 When $m_1 = m_2 = ... = m_q = 1$ problem (NSOCP) reduces to a nonlinear programming (NLP) and CRCQ (Definition 2.1) reduces to CRCQ in this case.

For any $y = (y_0, \widehat{y}) \in \mathbb{R} \times \mathbb{R}^{m-1}$, let $u_i(y) := \begin{cases} \frac{1}{2}\Big(1, (-1)^i \frac{\widehat{y}}{\|\widehat{y}\|}\Big), & \text{if } \widehat{y} \neq 0, \\ \frac{1}{2}\Big(1, (-1)^i w\Big), & \text{otherwise}, \end{cases}$
where $w \in \mathbb{R}^{m-1}$ can be any unitary vector, with $i \in \{1, 2\}$.

Now let us define, for any sets $J_B, J_-, J_+ \subseteq \{1, ..., q\}$ such that $\widehat{g}_j(x) \neq 0$ for every $j \in J_B$, the family of vectors

$$\mathcal{D}_{J_B,J_-,J_+}(x, w) := \Big\{Dg_j(x)^T u_1(g_j(x))\Big\}_{j\in J_B} \bigcup \Big\{Dg_j(x)^T(1, -w_j)\Big\}_{j\in J_-} \bigcup \Big\{Dg_j(x)^T(1, w_j)\Big\}_{j\in J_+}$$

where $w = [w_j]_{j\in J_-\cup J_+}$.

Definition 2.4 ([3, Definition 4.1]) We say that a feasible point $\bar{x}$ of (NSOCP) satisfies the *weak constant rank constraint qualification* (weak-CRCQ) if the following holds: For every sequence $\{x^k\}_{k\in\mathbb{N}} \to \bar{x}$, there exists some $I \subseteq_\infty \mathbb{N}$ and convergent eigenvector sequences

$$\{u_1(g_j(x^k))\}_{k\in I} \to \frac{1}{2}(1, -\bar{w}_j) \text{ and } \{u_2(g_j(x^k))\}_{k\in I} \to \frac{1}{2}(1, \bar{w}_j),$$

with $\|\bar{w}_j\| = 1$, $\forall j \in I_0(\bar{x})$, such that for all subsets $J_B \subseteq J_B(\bar{x})$ and $J_-, J_+ \subseteq J_0(\bar{x})$, we have that if the family of vectors $\mathcal{D}_{J_B,J_-,J_+}(\bar{x}, \bar{w})$ is linearly dependent, then $\mathcal{D}_{J_B,J_-,J_+}(x^k, w^k)$ is linearly dependent for all $k \in I$ large enough, where $\bar{w} = [\bar{w}_j]_{j\in J_-\cup J_+}$ and $w^k = [w^k_j]_{j\in J_-\cup J_+}$ satisfies

$$u_1(g_j(x^k)) = \frac{1}{2}(1, -w^k_j) \text{ and } u_2(g_j(x^k)) = \frac{1}{2}(1, w^k_j)$$

for each $j \in J_- \cup J_+$; where the notation $I \subseteq_\infty \mathbb{N}$ means that U is an infinite subset of $\mathbb{N}$.

Definition 2.5 ([3, Definition 5.1]) We say that a feasible point $\bar{x}$ of (NSOCP) satisfies the *sequential constant rank constraint qualification* (seq-CRCQ) if for all sequences $\{x^k\}_{k\in\mathbb{N}} \to \bar{x}$ and $\{\Delta^k_j\}_{k\in\mathbb{N}}$, $j \in I_0(\bar{x}) \cap I_B(\bar{x})$, such that $\Delta^k_j \to 0$ for every j, there exists some $I \subseteq_\infty \mathbb{N}$ and convergent eigenvector sequences

$$\{u_1(g_j(x^k) + \Delta^k_j)\}_{k\in I} \to \frac{1}{2}(1, -\bar{w}_j) \text{ and } \{u_2(g_j(x^k) + \Delta^k_j)\}_{k\in I} \to \frac{1}{2}(1, \bar{w}_j),$$

with $\|\bar{w}_j\| = 1$, $\forall j \in I_0(\bar{x})$, such that for all subsets $J_B \subseteq J_B(\bar{x})$ and $J_-, J_+ \subseteq J_0(\bar{x})$, we have if the family of vectors $\mathcal{D}_{J_B,J_-,J_+}(\bar{x}, \bar{w})$ is linearly dependent, then $\mathcal{D}_{J_B,J_-,J_+}(x^k, w^k)$

is linearly dependent for all $k \in I$ large enough, where $\bar{w} = [\bar{w}_j]_{j \in J_- \cup J_+}$ and $w^k = [w_j^k]_{j \in J_- \cup J_+}$ satisfies

$$u_1(g_j(x^k) + \Delta_j^k) = \frac{1}{2}(1, -w_j^k) \text{ and } u_2(g_j(x^k) + \Delta_j^k) = \frac{1}{2}(1, w_j^k)$$

for each $j \in J_- \cup J_+$.

Remark 2.6 *Definition 2.5 is basically Definition 2.4 with the addition of some perturbation sequences $\{\Delta_j^k\}_{k \in \mathbb{N}}$. Then, seq-CRCQ implies weak-CRCQ and Example 5.2 ([3]) shows that this implication is strict. However, Example 4.3 ([2]) shows that CRCQ does not imply Seq-CRCQ.*

3 Main results

In this section, we present the obtained results on the relationship between the CRCQ, weak-CRCQ and seq-CRCQ for NSOCP in case $q = 1$ and $m_1 = 2$.

Proposition 3.1 *Consider the NSOCP problem with $q = 1$ and $m_1 = 2$. Then CRCQ holds at feasible point $\bar{x}$ with $g(\bar{x}) = 0$ if and only if weak-CRCQ is also true there.*

Proof. Using [3, Remark 4.1], take any sequence $\{x^k\}_{k \in \mathbb{N}} \to \bar{x}$. We consider a partition of $\mathbb{N}$ as follows:
- $\mathcal{N}_0 := \{k \in \mathbb{N} : \widehat{g}(x^k) = 0\}$. For $k \in \mathcal{N}_0$, we can choose

$$u_1(g(x^k)) = \frac{1}{2}(1, -w^k) \text{ and } u_2(g(x^k)) = \frac{1}{2}(1, w^k)$$

for any w^k such that $\|w^k\| = 1$. When $\mathcal{N}_0$ is infinite, $\bar{x}$ satisfies the weak-CRCQ holds if the existence of a choice of $\{w^k\}_{k \in \mathcal{N}_0}$ with some convergent subsequence $\{w^k\}_{k \in I}, I \subseteq_\infty \mathcal{N}_0$, such that

$$Dg(\bar{x})^T(1, (-1)^i \bar{w}) = 0 \quad \text{only if} \quad Dg(x^k)^T(1, (-1)^i w^k) = 0$$

for all large $k \in I, i \in \{1, 2\}$; and, in addition, if $Dg(\bar{x})^T(1, -\bar{w})$ and $Dg(\bar{x})^T(1, \bar{w})$ are linearly dependent, then $Dg(x^k)^T(1, w^k)$ and $Dg(x^k)^T(1, -w^k)$ are linearly dependent, for every sufficiently large $k \in I$.
- $\mathcal{N}_1 := \{k \in \mathbb{N} : \widehat{g}(x^k) \neq 0\}$. This case is similar to the previous one, except that there is no freedom in the choice of w^k, as it is uniquely determined by $w^k = \widehat{g}(x^k)/\|\widehat{g}(x^k)\|$, for every $k \in \mathcal{N}_1$.

First, suppose that CRCQ holds at feasible point $\bar{x}$ with $g(\bar{x}) = 0$ and any sequence $\{x^k\}_{k \in \mathbb{N}} \to \bar{x}$.
When $\mathcal{N}_0$ is infinite, we can choose $w^k = 1 \in \mathbb{R}$ with $\|w^k\| = 1$. Then, we have $\bar{w} = 1$ and the rank of $\{Dg(\bar{x})^T(1, (-1)^i \bar{w})\}$, $\{Dg(x^k)^T(1, (-1)^i w^k)\}$ equal the dimension of

$Dg(\bar{x})^T[F_i^{\perp}]$, $Dg(x^k)^T[F_i^{\perp}]$, respectively, where $F_i = \{(v_1,v_2) \in \mathbb{R}^2 | v_1 = (-1)^{i+1}v_2 \geq 0\} \trianglerighteq K_2, i = 1,2$. Moreover, the rank of $\{Dg(\bar{x})^T(1,-\bar{w}),\ Dg(\bar{x})^T(1,\bar{w})\}$, $\{Dg(x^k)^T(1, - w^k)\ ,Dg(x^k)^T(1,w^k)\}$ equal the dimension of $Dg(\bar{x})^T[\mathbb{R}^2]$, $Dg(x^k)^T[\mathbb{R}^2]$, respectively, where $\mathbb{R}^2 = \{(0,0)\}^{\perp}$ with $\{(0,0)\} \trianglerighteq K_2$. Since CRCQ holds at $\bar{x}$, the dimension of $Dg(x)^T[F^{\perp}]$ remains constant for every x close enough to $\bar{x}$. Thus, the rank of $\{Dg(\bar{x})^T(1, (-1)^i\bar{w})\}$ equals the rank of $\{Dg(x^k)^T(1,(-1)^i w^k)\}$, $i = 1,2$, and the rank of $\{Dg(\bar{x})^T (1,-\bar{w}), Dg(\bar{x})^T(1,\bar{w})\}$ equals the rank of $\{Dg(x^k)^T(1,-w^k), Dg(x^k)^T(1,w^k)\}$ for all k large enough. This shows that weak-CRCQ holds at $\bar{x}$.
When $\mathcal{N}_1$ is finite, by replacing $\mathcal{N}_0$ by $\mathcal{N}_1$, we also get the desired conclusion.

Suppose that weak-CRCQ holds at $\bar{x}$ while CRCQ fails. Then, there exists some sequence $\{x^k\}_{k\in\mathbb{N}}$ and $F \trianglerighteq K_2$ such that the rank of $Dg(\bar{x})^T[F^{\perp}]$ is not equal the rank of $Dg(x^k)^T[F^{\perp}]$ for every $k \in \mathbb{N}$. We can assume, without loss of generality, that $\mathcal{N}_0$ is infinite, since weak-CRCQ holds at $\bar{x}$, the existence of a choice of $\{w^k\}_{k\in\mathbb{N}}$ with some convergent subsequence $\{w^k\}_{k\in I}, I \subseteq_{\infty} \mathbb{N}$ satisfy the Definition 2.4. Now, we assume that $w^k = 1$ for every $k \in I$, because $w^k \in \mathbb{R}$, $\|w^k\| = 1$ and the sequence $\{w^k\}_{k\in I}, I \subseteq_{\infty} \mathbb{N}$ is convergent. We have $\{w^k\}_{k\in I} \to \bar{w} = 1$.

Case 1. $F = F_1 = \{(v_1,v_2) \in \mathbb{R}^2 | v_1 = v_2 \geq 0\}$. Since the dimension of $Dg(\bar{x})^T[F_1^{\perp}]$ is not equal the dimension of $Dg(x^k)^T[F_1^{\perp}]$, then, the rank of $\{Dg(\bar{x})^T(1,-\bar{w})\}$ is not equal the rank of $\{Dg(x^k)^T(1,-w^k)\}$. By the continuity of $Dg(x)^T(1,-w)$ with respect to x, $\{Dg(\bar{x})^T(1,-\bar{w})\}$ has rank 0, and $\{Dg(x^k)^T(1,-w^k)\}$ has rank 1, contradicts weak-CRCQ at $\bar{x}$.

Case 2. $F = F_2 = \{(v_1,v_2) \in \mathbb{R}^2 | v_1 = -v_2 \geq 0\}$. By interchanging the roles of $-\bar{w}, -w^k$ and $\bar{w}, w^k$ the conclusion of Case 2 follows the one of Case 1.

Case 3. $F = F_3 = \{(0,0)\}, F^{\perp} = \mathbb{R}^2$. Since the dimension of $Dg(\bar{x})^T[\mathbb{R}^2]$ is not equal the dimension of $Dg(x^k)^T[\mathbb{R}^2]$ for every $k \in \mathbb{N}$, then, the rank of $\{Dg(\bar{x})^T(1,-\bar{w}),$ $\$Dg(\bar{x})^T(1,\bar{w})\}$ is not equal the rank of $\{Dg(x^k)^T(1,-w^k), Dg(x^k)^T(1,w^k)\}$ for every $k \in I$.

If $\text{rank}\{Dg(\bar{x})^T(1,-\bar{w}), Dg(\bar{x})^T(1,\bar{w})\} = 2$, by the continuity of $Dg(x)^T(1,-w)$ with respect to x, we have $\text{rank}\{Dg(x^k)^T(1,w^k), Dg(x^k)^T(1,w^k)\} = 2$ for all large $k \in I$, which contradicts the statement above.

If $\text{rank}\{Dg(\bar{x})^T(1,-\bar{w}), Dg(\bar{x})^T(1,\bar{w})\} = 1$, since weak-CRCQ holds at $\bar{x}$, $Dg(x^k)^T (1,-\bar{w})$ and $Dg(x^k)^T(1,\bar{w})$ are linearly dependent, for every sufficiently large $k \in I$. Thanks to the continuity of $Dg(x)^T(1,-w)$, we get $\text{rank}\{Dg(x^k)^T(1,w^k), Dg(x^k)^T (1,w^k)\} = 1$ for every sufficiently large $k \in I$, which contradicts the statement above.

If $\text{rank}\{Dg(\bar{x})^T(1,-\bar{w}), Dg(\bar{x})^T(1,\bar{w})\} = 0$, this means that $Dg(\bar{x})^T(1,(-1)^i\bar{w}) = 0$, $i = 1,2$. Since weak-CRCQ holds at $\bar{x}$, we have $Dg(x^k)^T(1,(-1)^i w^k) = 0$ for all large $k \in I, i = 1,2$. Thus, $\text{rank}\{Dg(x^k)^T(1,-\bar{w}), Dg(x^k)^T(1,\bar{w})\} = 0$ for every $k \in I$ sufficiently large, which contradicts the statement above.

Case 4. $F = F_4 = K_2, F^{\perp} = (0,0)$, which implies that the dimension of $Dg(\bar{x})^T[F^{\perp}]$ equals the dimension of $Dg(x^k)^T[F^{\perp}]$, that means CRCQ hold at $\bar{x}$.

Thus, weak-CRCQ holds at $\bar{x}$ implies CRCQ is also true there, completing the proof. □

The next proposition establishes relationships between seq-CRCQ and CRCQ for NSOCP in case $q = 1$ and $m_1 = 2$.

Proposition 3.2 *Consider the NSOCP problem with $q = 1$ and $m_1 = 2$. Then seq-CRCQ holds at feasible point $\bar{x}$ with $g(\bar{x}) = 0$ if and only if CRCQ is also true there.*

Proof. Using [3, Remark 5.1], seq-CRCQ holds at $\bar{x}$ if and only if, the dimension of

$$Dg(x)^T \text{span}(C_w^i) = \begin{cases} \text{span}(\{Dg(x)^T(1,w)\}), & \text{if } i = 1, \\ \text{span}(\{Dg(x)^T(1,-w), Dg(x)^T(1,w)\}), & \text{if } i = 2, \end{cases}$$

remains constant for every (x, w) close enough to $(\bar{x}, \bar{w})$, where $C_w^1 = \text{cone}(\{(1, w)\})$, for some $w \in \mathbb{R}$ such that $|w| = 1$, and $C_w^2 = \text{cone}(\{(1, -w), (1, w)\})$, for some $w \in \mathbb{R}$ such that $\|w\| = 1$.

Since $w, \bar{w} \in \mathbb{R}$ and $\|w\| = \|\bar{w}\| = 1$, then $w, \bar{w} \in \{-1, 1\}$. We have

$$\text{span}\Big(\Big\{Dg(x)^T\big(1, (-1)^i\big)\Big\}\Big) = Dg(x)^T[F_i^\perp]$$

where $F_i = \{(v_1, v_2) \in \mathbb{R}^2 | v_1 = (-1)^{i+1} v_2 \geq 0\} \trianglerighteq L_2,\ i = 1, 2$, and

$$\text{span}\Big(\big\{Dg(x)^T(1,-1), Dg(x)^T(1,1)\big\}\Big) = Dg(x)^T[\mathbb{R}^2] = Dg(x)^T[(0,0)^T]$$

with $\{(0,0)\} \trianglerighteq K_2$. Thus, the dimension of $Dg(x)^T\text{span}(C_w^i),\ i = 1, 2$ are constant for every (x, w) close enough to $(\bar{x}, \bar{w})$ if and only if, the dimension of $Dg(x)^T[F^\perp]$ remains constant for every x close enough to $\bar{x}$, $F \trianglerighteq K_2$ and completes the proof of the proposition. □

Combining Propositions 3.1, 3.2 and [4, Theorem 3.8], we obtain the following result.

Theorem 3.3 *Consider the NSOCP problem with $q = 1$ and $m_1 = 2$. Then CRCQ, weak-CRCQ, seq-CRCQ for the NSOCP problem coincide with CRCQ for the NLP problem.*

Proof. By Proposition 3.1, weak-CRCQ coincides with CRCQ for the NSCOP problem; by Proposition 3.2, seq-CRCQ coincides with CRCQ for the NSOCP problem; and thanks to [4, Theorem 3.8], CRCQ for the NSOCP coincides with CRCQ for the NLP problem. □

The following example exhibits the situation where the constraint qualification conditions in the Theorem 3.3 are satisfied.

Example 3.4 *Consider the constraint of (NSOCP)*

$$g(x) = (g_1(x), g_2(x)) := (-x_1 - x_2^2, 2x_1 + 2x_2^2) \in K_2$$

at $\bar{x} = (0, 0)$.

This constraint can be seen as the constraint of (NLP) problem as follows:

$$q_i(x) \leq 0, \quad i = 1, 2$$

where $q_1(x) := -g_1(x) + g_2(x) = 3x_1 + 3x_2^2$; $q_2(x) := -g_1(x) - g_2(x) = -x_1 - x_2^2$. *We have* $\nabla q_1(x) = (3,\ 6x_2)$, $\nabla q_2(x) = (-1,\ -2x_2)$ *and the index set of active inequality constraints* $\mathcal{I}(\bar{x}) = \{i \in \{1, 2\} | q_i(\bar{x}) = 0\} = \{1, 2\}$. *We see that for any index set* $\mathcal{A} \subseteq \mathcal{I}(\bar{x})$*, the system* $\{\nabla q_i(\bar{x}) | i \in \mathcal{A}\}$ *as the same rank for all* x *close enough to* $\bar{x}$. *This shows that CRCQ for the NLP problem holds at* $\bar{x}$. *By Theorem 3.3, CRCQ, weak-CRCQ, seq-CRCQ for the NSOCP problem also hold at* $\bar{x}$.

4 Concluding remarks

In this paper, we investigate the relationship between the constant rank-type constraint qualifications for nonlinear second-order cone programming. We obtained the equivalence of these constraint qualifications in the case of single constraint in two-dimensional space. An interesting topic is the comparison of these constraint qualifications in the general case, and with the metric subregularity constraint qualification will be of interest to us in the near future.

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TÓM TẮT

VỀ MỘT SỐ CHUẨN HÓA RÀNG BUỘC KIỂU HẠNG HẰNG CHO QUY HOẠCH NÓN BẬC HAI TRONG KHÔNG GIAN HAI CHIỀU

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Trong bài báo này, chúng tôi nghiên cứu mối quan hệ giữa một số chuẩn hóa ràng buộc kiểu hạng hằng cho bài toán quy hoạch nón bậc hai, được phát triển gần đây bởi R. Andreani và các cộng sự ([2],3]), trong không gian hai chiều.

Từ khoá: Chuẩn hóa rằng buộc kiểu hạng hằng; CRCQ; weak-CRCQ; seq-CRCQ; quy hoạch nón bậc hai phi tuyến; không gian hai chiêu.